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NUMERICAL METHODS

Conservative Schemes of Matter Transport in a System of Vessels Closed via the Heart

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Abstract— We suggest a conservative model and the corresponding conservative numerical method for analyzing the motion of matter in a branched vessel system closed via the heart. The efficiency of the algorithm is justified by test computations.

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INTRODUCTION

A broad range of problems in medicine, physiology, and pharmacology necessitate mathematical modeling of the transport of solutes in blood through the cardiovascular system. This problem was considered in quite a few papers, which schematically fall into two directions, modeling the local distribution of solutes in a separate vessel or its part and modeling the distribution of solutes over a network of vessels. As examples of papers in the latter direction, we note [1–10]. The present paper also belongs to this direction. When modeling the transport of components (gaseous, saline, hormone-containing, pharmacological, etc.) dissolved in blood over a network of vessels of complicated topology, one has to deal with specific features such as the small concentration of these solutes, their rapid propagation in the vessel system, the necessity of taking into account the interaction with tissues of the organism, and the closedness of the blood circulation system. As a rule, even small concentration of solutes can lead to essential reactions in the organism; in this connection, to obtain physiologically justified modeling, it is necessary that no "parasitic" sources of the considered solutes arise in the course of computations. Therefore, models and algorithms for solving these problems should have the property of conservativeness. The present paper deals with the solution of this problem. Here we suggest a conservative model and the corresponding conservative numerical method for analyzing the transport of solutes in an arbitrary graph of vessels, including the case in which the heart closing the vessel system is taken into account. The efficiency of the considered algorithm is justified by test computations.

1. INTEGRAL BALANCE RELATIONS FOR THE PROBLEM ON THE DIFFUSION TRANSPORT OF MATTERS

Following [8–13], to the circulatory system, we assign a graph of elastic vessels. The vessels are represented by the edges of the graph, and the vertices are either points where two or more vessels merge (branching points) or the heart, organs, and tissues. For any vessel system, vertices of degree 1 are said to be boundary, and the remaining vertices are said to be interior. The heart is represented by two boundary vertices corresponding to the heart ventricle–auricle and ventricle–aorta junctions. We consider the transport of solutes through the vessel system closed by the heart.

Following [10], we assume that the flow in each vessel is a quasi-one-dimensional. Therefore, the graph edges are directed segments. We assume that the blood density ϱ is constant. The pressure P in a vessel, the linear velocity U of blood flow, and the cross-section area S of a vessel are assumed to be known in the study of the matter transport problem. The above-mentioned variables are computed with the use of the software package CVSS [12] on the basis of hemodynamic equations [12–14].

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1.1. Matter Propagation in Blood Vessels

Consider a separate vessel. We assume that some matter with mass concentration $C(x,t)$ is distributed over the vessel. The matter propagates by convection and diffusion; i.e., the matter flux $W(x,t)$ is given by the formula [12; 15, p. 193]

$$W = W_c + W_d = CSU - DS\frac{\partial C}{\partial x}. \tag{1}$$

Throughout the following, the diffusion coefficient D is assumed to be one and the same constant in the entire vessel system.

Let us consider the vessel part $[x_1, x_2]$ and form a balance equation for the total matter mass on that part in the time interval $[t_1, t_2]$ (which corresponds to the volume balance for the case of constant density):

$$\int\limits_{x_1}^{x_2} (SC)|_{t_1}^{t_2}\,dx = \int\limits_{t_1}^{t_2} (W(x_1,t) - W(x_2,t))\,dt. \tag{2}$$

Consider a part of the vessel graph containing a branching vertex m and all of its incoming and outgoing edges $\{l_k\}_{k=1}^n$. The set of all edges with common vertex m is denoted by $\mathrm{Sh}(m)$, the subset of incoming edges by $\mathrm{Sh}^+(m)$, and the subset of outgoing edges by $\mathrm{Sh}^-(m)$.

Set

$$z_k^{(m)} = \begin{cases} -1 & \text{for} \quad l_k \in \mathrm{Sh}^-(m) \\ 1 & \text{for} \quad l_k \in \mathrm{Sh}^+(m). \end{cases} \tag{3}$$

On each edge l_k, we choose points $x_0^{(k)}$, $x_1^{(k)}$, and $x_2^{(k)}$ as shown in Fig. 1. The propagation of matter on each of the segments $[x_2^{(k)}, x_1^{(k)}]$ is described by the above-obtained relation (2). We form the balance equation for the matter volume on the union of the segments $\{[x_1^{(k)}, x_0^{(k)}]\}_{k=1}^n$,

$$\sum_{l_k\in \mathrm{Sh}(m)} z_k^{(m)} \int\limits_{x_1^{(k)}}^{x_0^{(k)}} (SC)|_{t_1}^{t_2}\,dx = \int\limits_{t_1}^{t_2} \sum_{l_k\in \mathrm{Sh}(m)} z_k^{(m)} W(x_1^{(k)},t)\,dt. \tag{4}$$

The balance of matter on a part of the vessel graph containing a vertex that corresponds to a tissue or an organ, is formed in a similar way:

$$\sum_{l_k\in \mathrm{Sh}(m)} z_k^{(m)} \int\limits_{x_1^{(k)}}^{x_0^{(k)}} (SC)|_{t_1}^{t_2}\,dx = \int\limits_{t_1}^{t_2} \left(G(m,t) + \sum_{l_k\in \mathrm{Sh}(m)} z_k^{(m)} W(x_1^{(k)},t) \right) dt; \tag{5}$$

here $G(m,t)$ is the matter source at the vertex m.

The obtained balance equations are based on the fact that the total volume of matter in the above-mentioned elements of the vessel graph can change owing to either fluxes on the boundary or local sources. Since any vessel graph consists of the above-considered elements, it follows that the total volume of matter in the absence of sources is preserved during the transport through an arbitrary graph of vessels with a possible change of that volume only owing to fluxes at the graph "input" or "output." Since we consider a vessel system closed by the heart, it follows that such fluxes are only fluxes on the auricle–heart and heart–aorta parts.

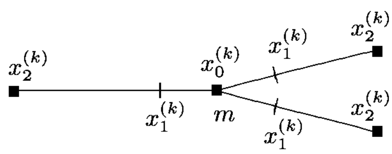

Fig. 1. A part of vessel graph with a branching vertex.

1.2. Matter Transport in the Heart

Following [10], we use the flux-coordinated heart model [12, 13]

$$V_H(t_2) = V_H(t_1) + \int_{t_1}^{t_2} (z_V (SU)_V(t) + z_A (SU)_A(t))\,dt,$$
$$C_H(t_2)V_H(t_2) = C_H(t_1)V_H(t_1) + \int_{t_1}^{t_2} (z_V W_V(t) + z_A W_A(t))\,dt; \qquad (6)$$

here V_H and C_H are the blood volume and the matter concentration, respectively, in the heart ventricle, $(SU)_V$ and W_V are the blood flux and the matter flux from the venous part of the blood system into the ventricle, which are zero during the systole period, and $(SU)_A$ and W_A are the fluxes from the ventricle into the aorta, which are zero during the diastole. The factors z_A and z_V are given by formulas (3) in the aorta and auricle, respectively.

The second relation in (6) closes the system of the balance equations (2), (4) and (5) written out on the graph edges.

2. DIFFERENCE APPROXIMATION TO THE INTEGRAL RELATIONS CONSERVATIVE DIFFERENCE SCHEMES

2.1. Difference Counterparts of the Balance Relations

Let the graph of the blood system consist of N vessels $\{l_k\}_{k=1}^N$. On each vessel l_k (of length L_k), we introduce the uniform grid $x_i^{(k)} = ih_k$, $i = 0, 1, \ldots, N_k$, $h_k = L_k/N_k$, and the uniform time grid with increment τ. We consider grid functions c, s, u, and w, which are the counterparts of the functions C, S, U, and W of continuous argument [16, p. 35]. In addition to the values $f_i^{(k)}$ of grid functions at the nodes $x_i^{(k)}$, we use the "half-integer" values

$$f_{i+1/2}^{(k)} = \frac{f_i^{(k)} + f_{i+1}^{(k)}}{2}, \qquad i = 0, \ldots, N_k - 1,$$

of all grid functions except for w, for which we set

$$w_{i+1/2}^{(k)} = \frac{c_i^{(k)} s_i^{(k)} u_i^{(k)} + c_{i+1}^{(k)} s_{i+1}^{(k)} u_{i+1}^{(k)}}{2} - D s_{i+1/2}^{(k)} \frac{c_{i+1}^{(k)} - c_i^{(k)}}{h_k}. \qquad (7)$$

We divide each vessel into segments $[x_0^{(k)}, x_{1/2}^{(k)}]$, $\{[x_{i-1/2}^{(k)}, x_{i+1/2}^{(k)}]\}_{i=1}^{N_k-1}$, and $[x_{N_k-1/2}^{(k)}, x_{N_k}^{(k)}]$. We write out the integral relation (2) for the interior segments $[x_{i-1/2}^{(k)}, x_{i+1/2}^{(k)}]$ in the form

$$\int_{x_{i-1/2}^{(k)}}^{x_{i+1/2}^{(k)}} (SC)|_{t_1}^{t_2}\,dx = \int_{t_1}^{t_2} (W(x_{i-1/2}^{(k)}, t) - W(x_{i+1/2}^{(k)}, t))\,dt.$$

We approximate this formula by the relation

$$h_k(s_i^{(k)} c_i^{(k)} - \check{s}_i^{(k)} \check{c}_i^{(k)}) = \tau(w_{i-1/2}^{(k)} - w_{i+1/2}^{(k)}), \qquad k = 1, \ldots, N, \quad i = 1, \ldots, N_k - 1, \qquad (8)$$

where the fluxes w are given by (7). Here and throughout the following, the notation $\check{f}_i^{(k)}$ corresponds to the values of the function f from the lower time layer t_1. The notation $f_i^{(k)}$ corresponds to the value of the function on the upper time layer t_2. Therefore, the unknowns are the $c_i^{(k)}$.

The segments of length $0.5h_k$ at the vessel ends are used to specify matter balance on the graph parts containing vertices.

Let m be a branching vertex, and let $\{l_k\}_{k=1}^n = \mathrm{Sh}(m)$. We use the quantities $z_k^{(m)}$ defined by relation (3), and in addition, we introduce the notation

$$E_k^{(m)} = \begin{cases} 0 & \text{for} \quad l_k \in \mathrm{Sh}^-(m) \\ N_k & \text{for} \quad l_k \in \mathrm{Sh}^+(m). \end{cases} \tag{9}$$

Then the segments $\{[x^{(k)}_{E_k^{(m)}-0.5z_k^{(m)}}, x^{(k)}_{E_k^{(m)}}]\}_{k=1}^n$ are adjacent to the vertex m. For these segments, we write out the integral equation (4) in the form

$$\sum_{l_k \in \mathrm{Sh}(m)} z_k^{(m)} \int\limits_{x^{(k)}_{E_k^{(m)}-0.5z_k^{(m)}}}^{x^{(k)}_{E_k^{(m)}}} (SC)|_{t_1}^{t_2}\, dx = \int\limits_{t_1}^{t_2} \sum_{l_k \in \mathrm{Sh}(m)} z_k^{(m)} W(x^{(k)}_{E_k^{(m)}-0.5z_k^{(m)}}, t)\, dt.$$

This relation can be approximated as follows:

$$\sum_{l_k \in \mathrm{Sh}(m)} \frac{1}{2} h_k (s^{(k)}_{E_k^{(m)}} c^{(k)}_{E_k^{(m)}} - \check{s}^{(k)}_{E_k^{(m)}} \check{c}^{(k)}_{E_k^{(m)}}) = \tau \Bigg(\sum_{l_k \in \mathrm{Sh}(m)} z_k^{(m)} w^{(k)}_{E_k^{(m)}-0.5z_k^{(m)}} \Bigg). \tag{10}$$

If m is a vertex corresponding to a tissue or an organ, then, by analogy with the last formula, we obtain the approximation

$$\sum_{l_k \in \mathrm{Sh}(m)} \frac{1}{2} h_k (s^{(k)}_{E_k^{(m)}} c^{(k)}_{E_k^{(m)}} - \check{s}^{(k)}_{E_k^{(m)}} \check{c}^{(k)}_{E_k^{(m)}}) = \tau \Bigg(g(m) + \sum_{l_k \in \mathrm{Sh}(m)} z_k^{(m)} w^{(k)}_{E_k^{(m)}-0.5z_k^{(m)}} \Bigg), \tag{11}$$

where $g(m) = G(m, t_2)$ is the volume density of matter sources or sinks at the vertex m.

If m corresponds to the heart, then there is a unique vessel $l_k \in \mathrm{Sh}(m)$. In addition, if the mitral valve is open and l_k communicates with a heart ventricle, then it follows from (2) that

$$z_k^{(m)} \int\limits_{x^{(k)}_{E_k^{(m)}-0.5z_k^{(m)}}}^{x^{(k)}_{E_k^{(m)}}} (SC)|_{t_1}^{t_2}\, dx = z_k^{(m)} \int\limits_{t_1}^{t_2} (W(x^{(k)}_{E_k^{(m)}-0.5z_k^{(m)}}, t) - W(x^{(k)}_{E_k^{(m)}}, t))\, dt.$$

In addition, by the second equation in (6), we have

$$(C_H V_H)|_{t_1}^{t_2} = \int\limits_{t_1}^{t_2} z_k^{(m)} W(x^{(k)}_{E_k^{(m)}}, t)\, dt.$$

If the mitral valve is closed, then the flux $W(x^{(k)}_{E_k^{(m)}}, t)$ is absent, and it follows from (2) that

$$z_k^{(m)} \int\limits_{x^{(k)}_{E_k^{(m)}-0.5z_k^{(m)}}}^{x^{(k)}_{E_k^{(m)}}} (SC)|_{t_1}^{t_2}\, dx = z_k^{(m)} \int\limits_{t_1}^{t_2} (W(x^{(k)}_{E_k^{(m)}-0.5z_k^{(m)}}, t))\, dt.$$

For the last three integral relations, we use the approximations

$$\frac{1}{2}h_k(s^{(k)}_{E^{(m)}_k}c^{(k)}_{E^{(m)}_k} - \check{s}^{(k)}_{E^{(m)}_k}\check{c}^{(k)}_{E^{(m)}_k}) = \tau z^{(m)}_k(w^{(k)}_{E^{(m)}_k-0.5z^{(m)}_k} - w^{(k)}_{E^{(m)}_k}), \tag{12}$$

$$c_H V_H - \check{c}_H \check{V}_H = \tau z^{(m)}_k w^{(k)}_{E^{(m)}_k}, \tag{13}$$

$$\frac{1}{2}h_k(s^{(k)}_{E^{(m)}_k}c^{(k)}_{E^{(m)}_k} - \check{s}^{(k)}_{E^{(m)}_k}\check{c}^{(k)}_{E^{(m)}_k}) = \tau z^{(m)}_k w^{(k)}_{E^{(m)}_k-0.5z^{(m)}_k}, \tag{14}$$

respectively, where $V_H = \check{V}_H + \tau z^{(m)}_k s^{(k)}_{E^{(m)}_k} u^{(k)}_{E^{(m)}_k}$ is the quantity obtained from the approximation to the first relation in (6).

For the quantity $w^{(k)}_{E^{(m)}_k}$ on the heart boundary, we use the expression

$$w^{(k)}_{E^{(m)}_k} = \begin{cases} c^{(k)}_{E^{(m)}_k} s^{(k)}_{E^{(m)}_k} u^{(k)}_{E^{(m)}_k} - D s^{(k)}_{E^{(m)}_k} z^{(m)}_k \frac{c_H - c^{(k)}_{E^{(m)}_k}}{h_k} & \text{if} \quad z^{(m)}_k u^{(k)}_{E^{(m)}_k} \ge 0 \\ c_H s^{(k)}_{E^{(m)}_k} u^{(k)}_{E^{(m)}_k} - D s^{(k)}_{E^{(m)}_k} z^{(m)}_k \frac{c_H - c^{(k)}_{E^{(m)}_k}}{h_k} & \text{if} \quad z^{(m)}_k u^{(k)}_{E^{(m)}_k} < 0. \end{cases} \tag{15}$$

Therefore, the expression for the convectional component of the flux depends on the flow direction. If the outflow from the heart takes place, then the quantity c_H is used, otherwise one uses the quantity $c^{(k)}_{E^{(m)}_k}$.

The obtained difference approximations to the integral balance equations should be supplemented with conditions at interior nodes of the graph. Prior to presenting particular examples of such relations, we derive an important corollary of the above-obtained equations (8), (10), and (12)–(14).

2.2. Conservation of the Total Volume of the Matter

Consider a single vessel l_k joining vertices m_1 and m_2. Let us sum relations (8) with respect to i for the interior segments of the vessel:

$$\sum_{i=1}^{N_k-1} h_k(s^{(k)}_i c^{(k)}_i - \check{s}^{(k)}_i \check{c}^{(k)}_i) = \sum_{i=1}^{N_k-1} \tau(w^{(k)}_{i-1/2} - w^{(k)}_{i+1/2}),$$

whence we have

$$\sum_{i=1}^{N_k-1} h_k(s^{(k)}_i c^{(k)}_i - \check{s}^{(k)}_i \check{c}^{(k)}_i) = \tau(w^{(k)}_{1/2} - w^{(k)}_{N_k-1/2})$$
$$= -\tau(z^{(m_1)}_k w^{(k)}_{E^{(m_1)}_k-0.5z^{(m_1)}_k} + z^{(m_2)}_k w^{(k)}_{E^{(m_2)}_k-0.5z^{(m_2)}_k}).$$

We have used notation (3) and (9) on the right-hand side in the last expression.

Let the vertices m_1 and m_2 correspond to the heart; moreover, let the mitral valve be open at the vertex m_2. Then condition (14) is satisfied on the segment adjacent to m_1. We add this relation to the above-represented equation,

$$\frac{h_k}{2}(s^{(k)}_{E^{(m_1)}_k}c^{(k)}_{E^{(m_1)}_k} - \check{s}^{(k)}_{E^{(m_1)}_k}\check{c}^{(k)}_{E^{(m_1)}_k}) + \sum_{i=1}^{N_k-1} h_k(s^{(k)}_i c^{(k)}_i - \check{s}^{(k)}_i \check{c}^{(k)}_i) = -\tau z^{(m_2)}_k w^{(k)}_{E^{(m_2)}_k-0.5z^{(m_2)}_k}.$$

As follows from the last formula, condition (14) is formed to as to ensure that the terms on the right-hand side corresponding to m_1 cancel out in the course of summation. By computing the sum of the last relation with relation (12) for the segment adjacent to m_2, we obtain

$$\frac{h_k}{2}(s^{(k)}_0 c^{(k)}_0 - \check{s}^{(k)}_0 \check{c}^{(k)}_0) + \sum_{i=1}^{N_k-1} h_k(s^{(k)}_i c^{(k)}_i - \check{s}^{(k)}_i \check{c}^{(k)}_i) + \frac{h_k}{2}(s^{(k)}_{N_k} c^{(k)}_{N_k} - \check{s}^{(k)}_{N_k} \check{c}^{(k)}_{N_k}) = -\tau z^{(m_2)}_k w^{(k)}_{E^{(m_2)}_k}.$$

By summing the resulting relation with relation (13) for the matter volume in the heart, we obtain

$$\frac{h_k}{2}(s_0^{(k)}c_0^{(k)} - \check{s}_0^{(k)}\check{c}_0^{(k)}) + \sum_{i=1}^{N_k-1} h_k(s_i^{(k)}c_i^{(k)} - \check{s}_i^{(k)}\check{c}_i^{(k)}) + \frac{h_k}{2}(s_{N_k}^{(k)}c_{N_k}^{(k)} - \check{s}_{N_k}^{(k)}\check{c}_{N_k}^{(k)}) + c_H V_H - \check{c}_H \check{V}_H = 0.$$

The last relation is the difference counterpart of the relation

$$\int\limits_0^{L_k} (SC)|_{t_1}^{t_2}\, dx + (C_H V_H)|_{t_1}^{t_2} = 0$$

and expresses the conservation of the total volume of the transported matter in the considered blood system.

Consider a branching vertex m of the vessel graph and the vessel set $\{l_k\}_{k=1}^{n} = \mathrm{Sh}(m)$. To be definite, we assume that the no-flow condition (14) is satisfied at the vessel endpoints opposite to m. Then for each l_k, we have

$$\frac{h_k}{2}(s^{(k)}_{N_k-E_k^{(m)}} c^{(k)}_{N_k-E_k^{(m)}} - \check{s}^{(k)}_{N_k-E_k^{(m)}} \check{c}^{(k)}_{N_k-E_k^{(m)}}) + \sum_{i=1}^{N_k-1} h_k(s_i^{(k)}c_i^{(k)} - \check{s}_i^{(k)}\check{c}_i^{(k)}) = -\tau z_k^{(m)} w^{(k)}_{E_k^{(m)}-0.5z_k^{(m)}}.$$

By summing the last relation with respect to k and by adding the resulting relation to condition (10), we obtain

$$\sum_{l_k\in \mathrm{Sh}(m)} \left(\frac{h_k}{2}(s_0^{(k)}c_0^{(k)} - \check{s}_0^{(k)}\check{c}_0^{(k)}) + \sum_{i=1}^{N_k-1} h_k(s_i^{(k)}c_i^{(k)} - \check{s}_i^{(k)}\check{c}_i^{(k)}) + \frac{h_k}{2}(s_{N_k}^{(k)}c_{N_k}^{(k)} - \check{s}_{N_k}^{(k)}\check{c}_{N_k}^{(k)})\right) = 0.$$

Therefore, in the absence of external fluxes, the total volume of the matter in the considered system is conserved.

In a similar way, one can show that on a part of the vessel graph containing a vertex-tissue, the change of the matter volume can take place owing to either external fluxes or matter sources at the vertex.

We generalize the above-obtained property to an arbitrary graph of N vessels l_k closed by the heart:

$$\sum_{k=1}^{N}\left(\frac{h_k}{2}(s_0^{(k)}c_0^{(k)} - \check{s}_0^{(k)}\check{c}_0^{(k)}) + \sum_{i=1}^{N_k-1} h_k(s_i^{(k)}c_i^{(k)} - \check{s}_i^{(k)}\check{c}_i^{(k)}) + \frac{h_k}{2}(s_{N_k}^{(k)}c_{N_k}^{(k)} - \check{s}_{N_k}^{(k)}\check{c}_{N_k}^{(k)})\right) + c_H V_H - \check{c}_H \check{V}_H = 0.$$

This relation holds in view of condition (8) on the interior segments of the vessels, conditions (11) and (11) at the internal vertices of the graph, and relations (12)–(14) on the boundary with the heart. Therefore, for any difference scheme including the above-mentioned conditions, the total volume of matter in the blood system is preserved,

$$\sum_{k=1}^{N}\left(\frac{h_k}{2}s_0^{(k)}c_0^{(k)} + \sum_{i=1}^{N_k-1} h_k s_i^{(k)}c_i^{(k)} + \frac{h_k}{2}s_{N_k}^{(k)}c_{N_k}^{(k)}\right) + c_H V_H = \text{const}. \qquad (16)$$

2.3. Relations at Interior Vertices. Examples of Conservative Systems

The validity of conditions (8), (10), and (12)–(14) provides the conservativeness of an arbitrary difference scheme including above-mentioned equations. As was mentioned above, the balance relations should be supplemented by relations at the interior nodes of the graph. Indeed, if n edges

l_k enter a vertex m, then one should write out n relations for the n unknowns $c^{(k)}_{E^{(m)}_k}$. Equation (10) is one of them. By way of example, consider two possibilities of specifying additional relations.

C-scheme. The C-scheme is the system of difference equations obtained by supplementing the balance equations with the following relations at the interior nodes:

$$c^{(k)}_{E^{(m)}_k} = c^{(i)}_{E^{(m)}_i}, \qquad k = 1, \dots, n, \qquad k \neq i. \tag{17}$$

SC-scheme. The SC-scheme is the system of difference equations obtained by supplementing the balance equations with the following relations at the interior nodes:

$$s^{(k)}_{E^{(m)}_k} c^{(k)}_{E^{(m)}_k} = s^{(i)}_{E^{(m)}_i} c^{(i)}_{E^{(m)}_i}, \qquad k = 1, \dots, n, \qquad k \neq i. \tag{18}$$

All presented difference schemes have the property (16), i.e., are conservative.

3. TEST COMPUTATIONS

Computations were performed with the use of the software package CVSS (CardioVascular Simulating System, the software package for the computation of hemodynamic flows developed by the Department of Computational Methods at the Faculty of Computational Mathematics and Cybernetics at Moscow State University) [12]. We have used the vessel graph of greater circulation (Fig. 2) described in the cited paper. Edge 1 of the graph corresponds to the aorta, and edge 55 corresponds to the venous sinus. The boundary vertices of the graph incident to these edges correspond to the heart.

The heart is represented by a flux-coordinated model with the following parameters: the time of the heart cycle is $\tau_d = 0.8$ sec, the systole time is $\tau_s = 0.25$ sec, the minimum and maximum volumes of the blood in the ventricle are $V_{\min} = 50$ ml and $V_{\max} = 150$ ml, the initial blood volume is $V_{\text{init}} = 140$ ml, and the shock volume of the ventricle is $V_s = 70$ ml.

The zero initial conditions for the concentration C are posed everywhere on the graph except for the vessels 1 and 55 on which

$$C(x^{(k)}, 0) = \frac{1}{2}\left[\cos\left(\pi\left(2\frac{x^{(k)}}{L_k} - 1\right)\right) + 1\right], \qquad x^{(k)} \in [0, L_k], \qquad k = 1, 55. \tag{19}$$

The definition of a nonzero initial perturbation in the auricle and aorta permits one to trace the transport of matter through the heart and the propagation of the matter from the arterial part of the vessel system to the venous one.

In computations, the diffusion constant D was considered to be equal to 10 cm^2/sec.

3.1. C-Scheme

We have used the implicit version of the C-scheme (8), (10), (12)–(15), (17). To illustrate the process of matter propagation over the vessel system, we have chosen the path 1–38–56–67–78–79–34–32–33–55. The first four edges in this path correspond to the arterial part, and the remaining edges correspond to the venous one.

Figure 3 presents the concentration profiles in the above-mentioned chain of vessels at various time instants. The upper part of the graph represents the scheme of the considered path, which provides the correspondence between segments of the abscissa axis and the vessels. The notation for the vertices corresponds to Fig. 2. The vertical lines in the figure imply that an edge that does not belong to the considered path enters a branching vertex or exits from it.

Curve *1* corresponds to the initial time instant, and the concentrations (19) are given in the aorta and the auricle. Curve *2* represents the passage of matter through a branching vertex of the graph. In this case, after passing vessel 38, the matter gets mainly into vessel 37 (it does not belong to the path) and only to a smaller extent into the considered vessel 56. At that time, the matter from

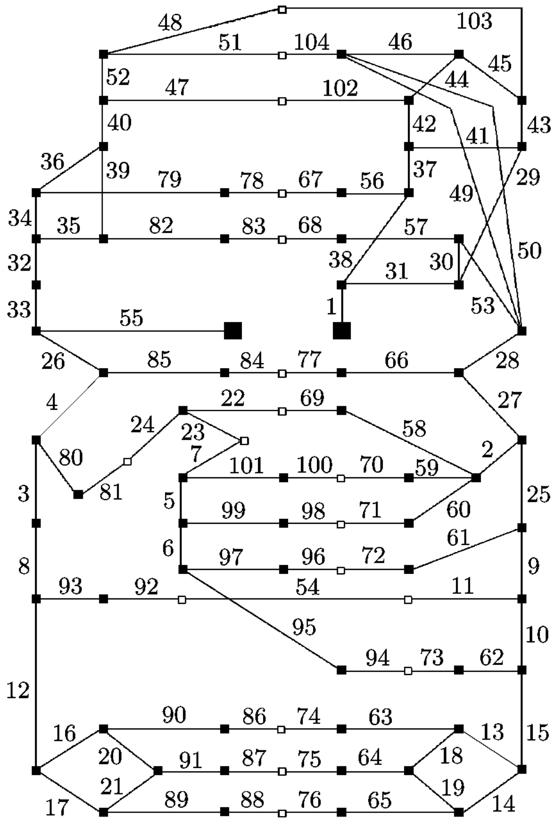

Fig. 2. Graph of vessels of the greater circulation; □ are interior branching vertices, and • are vertices corresponding to separate organs and tissues.

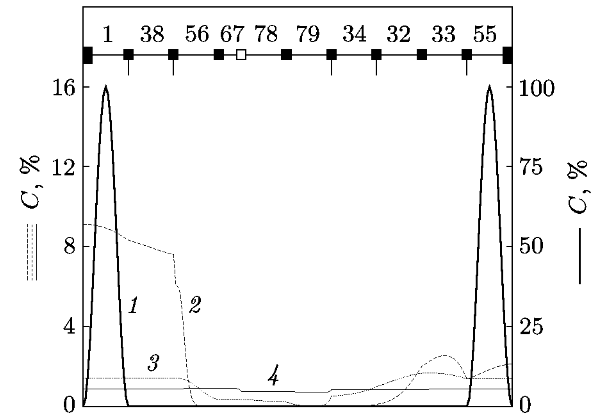

Fig. 3. The concentration profile. The C-scheme.

the venous part of the system passes through the heart and gets into the aorta. Curve *3* shows that the transported substance propagates through different parts of the vessel network at different velocities. For example, the matter gets into vessel 34 after passing the path 37–42–102–47–40–36 much more rapidly than passing through the illustrated path. The smaller transport velocity on the considered path is caused by the presence of a resistive vessel (vessel 67), while the path 37–42–102–47–40–36 contains no resistive vessels. Curve *3* also illustrates the passage of matter through

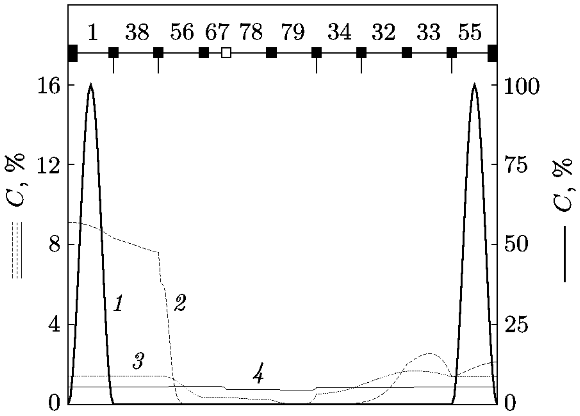

Fig. 4. Total matter volumes in the circulatory system. The C-scheme.

the venous part of the vessel graph and its second entry into the auricle. Achieving the heart, the matter is accumulated in the ventricle and is ejected again in the arterial part, after which a new wave of concentrations passes through vessels from the aorta to the auricle. In the course of time, owing to the matter diffusion, a constant concentration of the matter is stabilized in the whole circulatory system (the curve *4*). The time of achieving a constant concentration is 45–50 sec.

The described dynamics corresponds to physiological laws of the process of diffusion transport of matter through the circulatory system.

Note that, during the computations, the circulatory system remains closed with respect to the total volume of the matter. Figure 4 represent the time dependencies of the matter volumes in the heart, vessels, and the system in the whole. Curves corresponding to the first two volumes have oscillations of two types [10], but the total volume of matter is preserved in the system. Therefore, the computations justify that the C-scheme has the conservativeness property (16).

The performed computations showed that the C-scheme is a conservative algorithm for the problem on the matter transport through a system of vessels closed by the heart, which adequately represents the matter transport through all parts of the graph of vessels: inside each vessel, through branching vertices, tissues, and through the operating heart.

3.2. The SC-Scheme

We tested an implicit version of the SC-scheme (8), (10), (12)–(15), (18). Computations showed that, like the C-scheme, the SC-scheme preserves the volume of the matter transported in the system, i.e., is a conservative algorithm.

However, conditions (18) have an undesired effect on the solution. The vertices generate instability and lead to a critical growth of the error and solution oscillations. Possibly, this is related to the fact that conditions (18) do not provide the continuity of the concentration at the vertices. Cross-sections of some adjacent vessels can essentially differ. For example, as a rule, at vertices-tissues at which resistive arterial and venous vessels join, the effective [13] cross-sections can differ by 2–3 orders of magnitude. This can result in substantial discontinuities of the solution.

Therefore, in spite of the algorithm conservativeness, computations in accordance with the SC-scheme do not adequately reconstruct the process of diffusion transport of matter through the circulatory system. It follows that the choice of additional relations for the concentration at the interior nodes of the graph can make a substantial influence on the numerical solution.

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