Oscillatory Flow in Arteries: the Constrained Elastic Tube as a Model of Arterial Flow and Pulse Transmission

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Oscillatory Flow in Arteries: the Constrained Elastic Tube as a Model of Arterial Flow and Pulse Transmission

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§ 1. INTRODUCTION

THE study of the propagation of pressure waves in a viscous liquid contained in an elastic tube goes back at least as far as the work of Witzig (1914). In a recent paper, Morgan and Kiely (1954) made a number of shrewd and pertinent criticisms of earlier work, and gave two approximate analytical formulae for the pulse-velocity for two sets of limiting conditions, characterized by them as liquids of 'small' and 'large' viscosity.

In an independent attack on this problem (Womersley 1955 a), the author showed that the variation in pulse-velocity with frequency and viscosity can be expressed as a function of a single non-dimensional parameter, α, where

$$\alpha = R\sqrt{n/\nu}$$

R being the radius of the tube, n the circular frequency (i.e. the frequency in cycles per second multiplied by 2π) and ν the kinematic viscosity of the liquid. The variation in pulse-velocity with α was computed from $\alpha=1$ to $\alpha=10$ by steps of 0·05, together with the damping coefficient, and the quantities required for the calculation of the rate of flow from the pressure gradient, and tables of these quantities have been prepared by the Computation Laboratory of Harvard University (Womersley 1957). It is the purpose of this paper to show that by the modification of a single parameter (the ratio of the wall thickness to the radius of the tube) the equations of Womersley (1955 a) can be used to describe the motion when the tube is 'loaded' with added mass which increases its inertia without taking part in the elastic deformation, and 'tethered' by a longitudinal elastic constraint. If this constraint is very stiff, i.e. if it has a natural frequency that is high compared with the pulse frequency, the solution for this limiting condition is a particularly simple one. The relationship between pressure gradient and rate of flow reduces to that for a rigid tube, but the pulse velocity remains finite and its variation with frequency takes a very simple form. It is also shown that some experimental results obtained by Lawton and Greene (1956) are in fair agreement with the assumption that the limiting condition of stiff constraint applies in the abdominal aorta. This evidence, taken together with the agreement between measurements of flow made by McDonald (1955) and the formula for oscillatory flow in a rigid tube, indicates that this simple theory is adequate.

§ 2. THE 'FREQUENCY EQUATION' FOR THE PULSE-VELOCITY

The equations of motion of the tube with added mass, but without longitudinal constraint, were given by Morgan and Ferrante (1955) but were not discussed in detail by them. If the thickness of the tube is h, ρ its density, and h_1, R_1, ρ_1, are the thickness, mean radius, and density of the surrounding mass, Morgan and Ferrante state that the thickness, h, is to be replaced in the original equations of motion by a quantity H' where

$$\frac{H'}{h}=\left(1+\frac{h_1}{h}\cdot\frac{\rho_1 R_1}{\rho R}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If, in addition to this mass-loading, there is an elastic constraint which affects the longitudinal motion of the wall, and that alone, the equation of motion for the longitudinal displacement is

$$\ddot{\zeta}+m^2\zeta=-\frac{\rho_0}{\rho}\cdot\frac{\nu}{hR}\left(\frac{\partial w}{\partial y}\right)_{y=1}+\frac{Bh}{H'\rho}\left(\frac{\partial^2\zeta}{\partial z^2}+\frac{\sigma}{R}\cdot\frac{\partial\xi}{\partial z}\right) \quad . \quad . \quad (2)$$

in which $m/2\pi$ is the natural frequency of the longitudinal constraint. Equation (2) above replaces eqn. (16) of Womersley (1955 a). The notation is the same as in that account, viz.

ζ=longitudinal displacement of the wall ; ξ=radial displacement of the wall ; $B=E/(1-\sigma^2)$, E=Young's modulus, σ=Poisson's ratio ; w=longitudinal velocity of the liquid ; and y=non-dimensional radial coordinate$=r/R$.

If now eqn. (2) above replaces eqn. (16) of Womersley (1955 a), and the resulting set of equations is solved in the same manner, the form of the frequency-equation for the wave-velocity is unchanged. It is

$$(1-\sigma^2)x^2+2Gx+H=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where

$$x=kB/\rho c^2, \quad c=\text{wave-velocity}$$

$$G=\frac{1+\frac{1}{4}-\sigma}{1-F_{10}}+\tfrac{1}{2}K+\sigma-\tfrac{1}{4}$$

$$H=\frac{1+2K}{1-F_{10}}-1$$

and

$$1-F_{10}=1-\frac{2J_1(\alpha i^{3/2})}{\alpha i^{3/2}J_0(\alpha i^{3/2})}$$

the function which appears in the simple theory of oscillatory flow in a rigid tube, and whose modulus and phase have already been tabulated (Womersley 1955 b). Equation (3) has exactly the same form as eqn. (39) of Womersley (1955 a). The only difference between them is in the definition of K. For the tube with added mass and longitudinal constraint

$$K=\left(1+\frac{h_1}{h}\cdot\frac{\rho_1 R_1}{\rho R}\right)\left(1-\frac{m^2}{n^2}\right). \quad . \quad . \quad . \quad . \quad . \quad (4)$$

A number of interesting conclusions follow almost immediately from (4). If the frequency of the oscillation is the same as the natural frequency of the constraint ($m=n$), $K=0$ and the tube behaves as though it had zero mass. If $m>n$, K is negative, and if $m \gg n$, i.e. if the constraint is very stiff, and if the mass-factor is large, $K \rightarrow -\infty$ and eqn. (3) reduces to

$$-x+\frac{2}{1-F_{10}}=0 \qquad (5)$$

i.e.

$$x=\frac{2}{1-F_{10}}. \qquad (6)$$

We have, therefore,

$$(1-\sigma^2)\tfrac{1}{2}x=\frac{1-\sigma^2}{1-F_{10}}. \qquad (7)$$

Defining the pulse-velocity for a liquid of zero viscosity as c_0, we have

$$c_0=\left\{\frac{h}{2R}\left(1+\frac{h_1}{h}.\frac{\rho_1 R_1}{\rho R}\right).\frac{E}{\rho_0}\right\}^{1/2} \qquad (8)$$

and

$$\frac{c_0}{c}=\sqrt{\left(\frac{1-\sigma^2}{1-F_{10}}\right)}. \qquad (9)$$

From eqn. (48) of Womersley (1955 a) the motion of the liquid is given by

$$w=\frac{A_1}{\rho_0 c}\left\{1+\eta\frac{J_0(\alpha i^{3/2}y)}{J_0(\alpha i^{3/2})}\right\}\exp(int) \qquad (10)$$

the formula for η being (eqn. (47) of the same)

$$\eta=\frac{2}{x}.\frac{1}{F_{10}-2\sigma}-\frac{1-2\sigma}{F_{10}-2\sigma}. \qquad (11)$$

Inserting the value of x from (6) this becomes

$$\eta=\frac{1-F_{10}}{F_{10}-2\sigma}-\frac{1-2\sigma}{F_{10}-2\sigma}=-1$$

so that (10) becomes

$$w=\frac{A_1}{\rho_0 c}\left\{1-\frac{J_0(\alpha y i^{3/2})}{J_0(\alpha i^{3/2})}\right\}\exp(int) \qquad (12)$$

In terms of the pressure gradient this may be written

$$w=\frac{A_1}{in\rho_0}\left\{1-\frac{J_0(\alpha y i^{3/2})}{J_0(\alpha i^{3/2})}\right\}\exp(int)$$

which is the same as the form for the rigid tube (Womersley 1955 b). This follows from the fact that if

$$p=A_1\exp\{in(t-z/c)\}$$

$$-\frac{\partial p}{\partial z}=\frac{in}{c}.A_1.\exp\{in(t-z/c)\}$$

and therefore if

$$-\frac{\partial p}{\partial z}=A_1.\exp\{in(t-z/c)\}$$

$$A_1=\frac{c}{in}A_1'. \qquad (13)$$

It would be expected, from physical considerations, that if the longitudinal constraint were stiff enough to inhibit the longitudinal motion of the wall, the longitudinal velocity of the liquid would have to reduce to zero at the wall. The fact that $\eta \to -1$ as $K \to -\infty$ is, therefore a check on the accuracy of the analysis.

If, following the notation of the previous paper (Womersley 1955 a) we write

$$c_0/c = X - iY$$

the wave velocity, c_1, is given by

$$c_1 = c_0/X$$

and the damping-coefficient by $2\pi Y/X$, the amplitude being reduced in the ratio $\exp(-2\pi Y/X)$ for each wavelength of travel.

In fig. 1 the variation of c_1/c_0 with α is shown for $\sigma = \frac{1}{2}$, $K = 0$, -2, and fig. 2 shows similar graphs for $\exp(-2\pi Y/X)$

In the limiting condition of stiff constraint,

$$\frac{c_0}{c} = \frac{\sqrt{(1-\sigma^2)}}{(M'_{10})^{1/2}} \exp(-i\tfrac{1}{2}\epsilon'_{10}) \quad . \quad . \quad . \quad . \quad . \quad (14)$$

the quantities M'_{10} and ϵ'_{10} having been previously tabulated (Womersley 1955 b). For these conditions, therefore,

$$\frac{c_1}{c_0} = \frac{(M'_{10})^{1/2}}{\sqrt{(1-\sigma^2)}} \,.\, \sec \tfrac{1}{2}\epsilon'_{10} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and

$$\frac{2\pi Y}{X} = \tan \tfrac{1}{2}\epsilon'_{10}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

It will be noted that the ratio c_1/c_0 does not tend to unity as $\alpha \to \infty$. c_0 is not the corresponding wave-velocity when $\alpha \to \infty$, but the limiting value for the same tube without constraint, and this increase in c_1/c_0 shows the stiffening effect of the longitudinal constraint. Comparison with the corresponding quantities for the freely-moving tube with $\sigma = \frac{1}{2}$ shows that for the constrained tube the amount of damping in transmission is increased. This might at first sight seem to be an argument against the constrained tube as an arterial model. Moreover, the 'peaking' effect on the pulse-wave due to variation in wave velocity with frequency will also be small, as may be seen from fig. 1. In fact the rise in pulse-pressure during transmission is caused by reflections at junctions (Womersley, in preparation), and the attenuation due to viscosity is no bar to the acceptability of the constrained tube as a model of an artery.

§ 3. Pressure-Flow and Pressure-Diameter Relationships

From the point of view of physical principles, the motion of the liquid is best understood in terms of the dependence of flow on the pressure-gradient. The analogy between pressure-gradient and voltage, and between rate of flow and current, leads to the concept of fluid impedance.

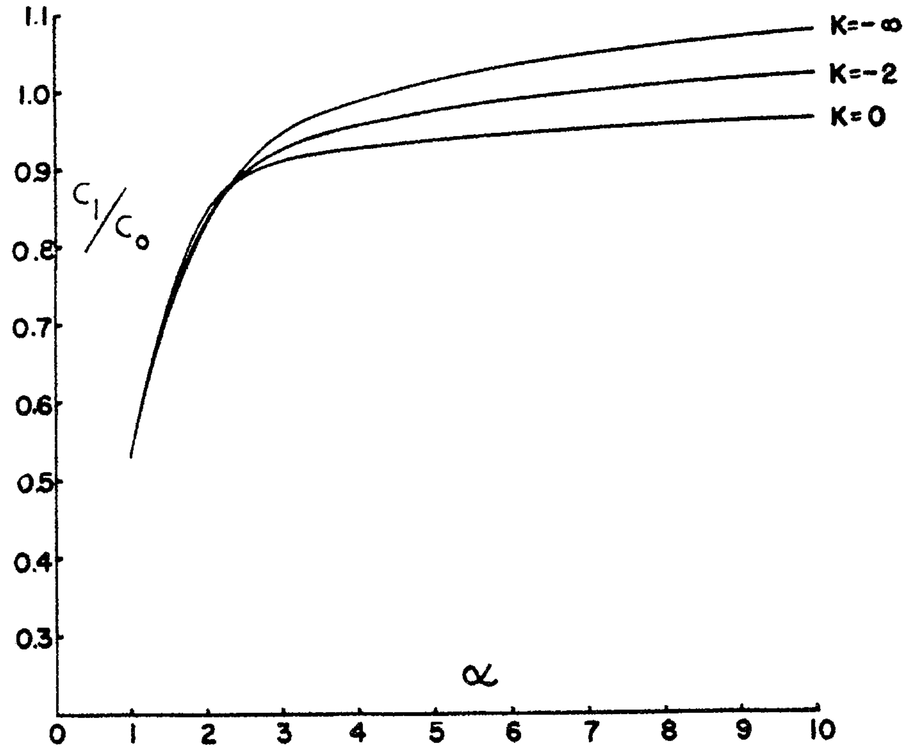
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
C1/C0
K=-∞
K=-2
K=0
α
0
1
2
3
4
5
6
7
8
9
10

Fig. 1

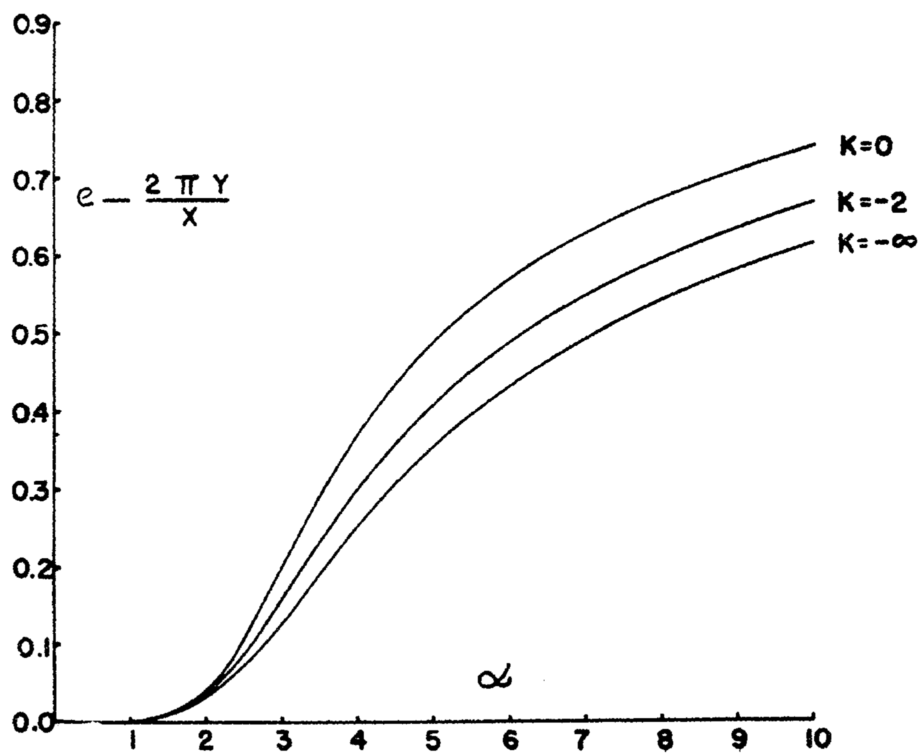
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
e — 2 π Y / X
K=0
K=-2
K=-∞
α
1
2
3
4
5
6
7
8
9
10

Fig. 2

However, if the pressure-gradient is created by a travelling wave, there is a simple relationship between the pressure gradient and the pressure itself, and the rate of flow can be expressed in terms of the pressure.

We have

$$\bar{w}=\frac{p}{\rho c}\,.\,M''_{10}\exp\,(i\epsilon''_{10}) \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where it is to be noted that in this formula c must be in its complex form. Since $c_0/c=X-iY$,

$$\bar{w}=\frac{p}{\rho c_0}(X-iY)\,.\,M''_{10}\exp\,(i\epsilon''_{10}) \qquad . \quad . \quad . \quad . \quad (18)$$

so that the effect of the damping of the wave in transmission is to reduce the phase advance of flow over pressure. In the limiting condition of stiff constraint,

$$\bar{w}=\frac{p}{\rho c_0}\,\tfrac{1}{2}\sqrt{3}M'_{10}\exp\,(\tfrac{1}{2}i\epsilon'_{10}) \qquad . \quad . \quad . \quad . \quad . \quad (19)$$

so that in these circumstances the maximum phase-lead of flow over pressure will be 45°.

These formulae no longer hold if there is any appreciable reflected wave. For consider

$$p=A_1\exp\,\{in(t-z/c)\}+A_2\exp\,\{in(t+z/c)\}. \quad . \quad . \quad . \quad (20)$$

The formula for the average velocity will now be

$$\bar{w}=\left\{\frac{A_1}{\rho c}\,.\,\exp\,\{in(t-z/c)\}-\frac{A_2}{\rho c}\,.\,\exp\,\{in(t+z/c)\}\right\}M''_{10}\exp i\epsilon''_{10}$$

and therefore, at a particular value of z, say at $z=0$,

$$\bar{w}=\frac{c}{\rho c}\,.\,\frac{A_1-A_2}{A_1+A_2}\,.\,M''_{10}\exp\,(i\epsilon''_{10}). \qquad . \quad . \quad . \quad . \quad (21)$$

This shows that the rate of flow cannot be calculated from a single pulse-pressure measurement unless the relative phase and amplitude of any reflected wave that may be present is also known. This does not apply to the relationship between pressure-gradient and flow.

In the previous paper (Womersley 1955 a) it was shown that a simple relationship exists between radial expansion and the average velocity across the tube at any instant. It is

$$\frac{2\xi}{R}=\frac{\bar{w}}{c} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where it is to be noted that c must be in its complex form. This relationship can be deduced very simply from the equation of continuity. For, if w is the longitudinal velocity and u the radial velocity the equation of continuity is

$$\frac{1}{r}\,.\,\frac{\partial}{\partial r}(ru)+\frac{\partial w}{\partial z}=0$$

and if $u=u_1 \exp \{in(t-z/c)\}$, $w=w_1 \exp \{in(t-z/c)\}$ then

$$\frac{1}{r}\cdot\frac{\partial}{\partial r}(ru_1)=in\frac{w_1}{c}$$

and, therefore, writing $r/R=y$ and integrating from $y=0$ to $y=1$,

$$[u_1]_{y=1}=\frac{inR}{2c}\cdot\bar{w}$$

But at $y=1$, $u=\frac{\partial\xi}{\partial t}$ and therefore

$$\frac{2\xi}{R}=\frac{\bar{w}}{c} \quad \ldots\ldots\ldots\ldots \quad (23)$$

If the formula for the average velocity in terms of the pressure is inserted, this becomes

$$\frac{2\xi}{R}=\frac{p}{\rho c^2}\cdot M''_{10}\exp i\epsilon''_{10}=\frac{p}{\rho c_0^2}\left(\frac{c_0}{c}\right)^2 M''_{10}\exp(i\epsilon''_{10}). \quad . \quad (24)$$

This may also be written

$$\frac{2\xi}{R}=\frac{p}{\rho c_0^2}(1-\sigma^2)\tfrac{1}{2}x(1+\eta F_{10}) \quad \ldots\ldots \quad (25)$$

For all finite values of K, the phase of the complex quantity

$$x(1+F_{10})$$

is positive, so that expansion always leads pressure.

For the condition of limiting constraint this takes a particularly simple form. When $K\to-\infty$, $\eta\to-1$, and $\frac{1}{2}x\to 1/1-F_{10}$, so that

$$\frac{2\xi}{R}=(1-\sigma^2)\frac{p}{\rho c_0^2}$$

and when $\sigma=\frac{1}{2}$,

$$\frac{2\xi}{R}=\frac{3}{4}\cdot\frac{p}{\rho c_0^2}. \quad \ldots\ldots\ldots\ldots \quad (26)$$

Pressure and expansion will be in phase at all frequencies.

These formulae connecting the harmonic components of the variations in pressure with the corresponding variations in diameter take the same form when there is a reflected wave present.

For, if

$$p=A_1\exp\{in(t-z/c)\}+A_2\exp\{in(t+z/c)\}$$

$$\bar{w}=\left\{\frac{A_1}{\rho c}\exp[in(t-z/c)]-\frac{A_2}{\rho c}\exp[in(t+z/c)]\right\}M''_{10}\exp(i\epsilon''_{10})$$

and

$$\frac{-\partial\bar{w}}{\partial z}=in\left\{\frac{A_1}{\rho c^2}\exp[in(t-z/c)]+\frac{A_2}{\rho c^2}\exp[in(t+z/c)]\right\}M''_{10}\exp(i\epsilon''_{10})$$

$$=in\cdot\frac{p}{\rho c^2}\cdot M''_{10}\exp(i\epsilon''_{10})$$

and on substitution in the equation of continuity, eqn. (26) will be obtained, as before.

An experimental confirmation of (26) can be obtained from some results recently published by Lawton and Greene (1956), two sets of which are shown in figs. 3 and 4. The observed points are joined by straight lines. The circles are points on a four-harmonic Fourier series fitted to the observations. The close coincidence in phase between pressure and diameter is immediately evident to the eye. The table gives the phase-lag of diameter behind pressure for the separate harmonics of the two curves.

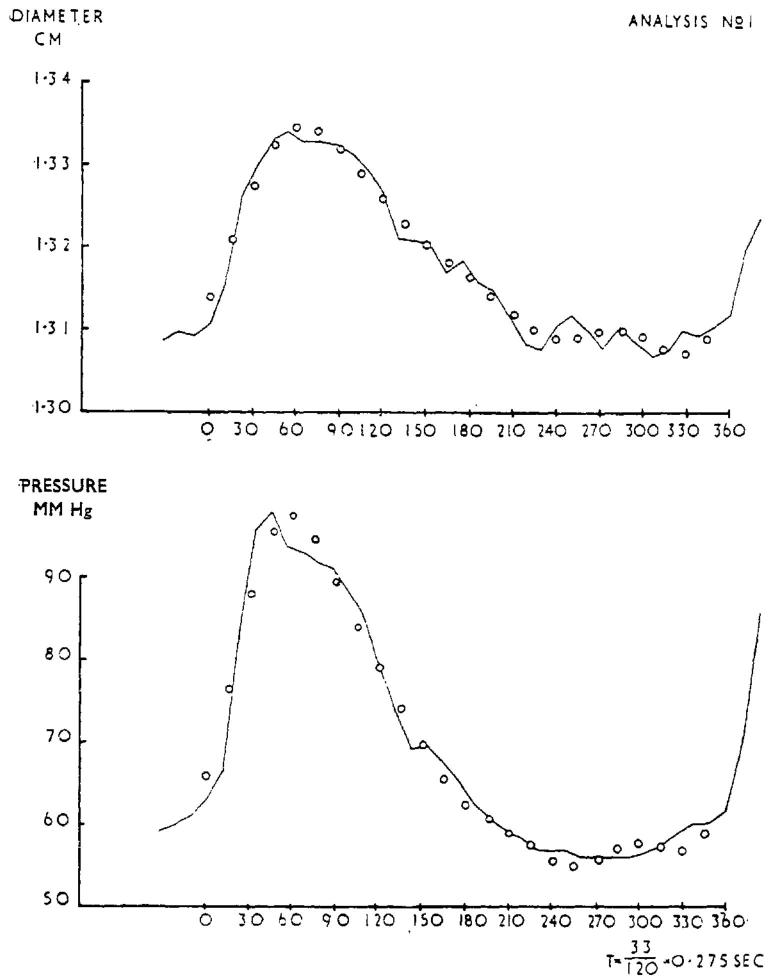

Fig. 3

The amplitude of the third and fourth harmonics was small, that of the third harmonic being a little less than one-sixth of that of the fundamental, that of the fourth harmonic about 5%. It would seem that until measurements of greater accuracy become available, the simplest form of the theory (i.e. $K = -\infty$) will be reasonably adequate.

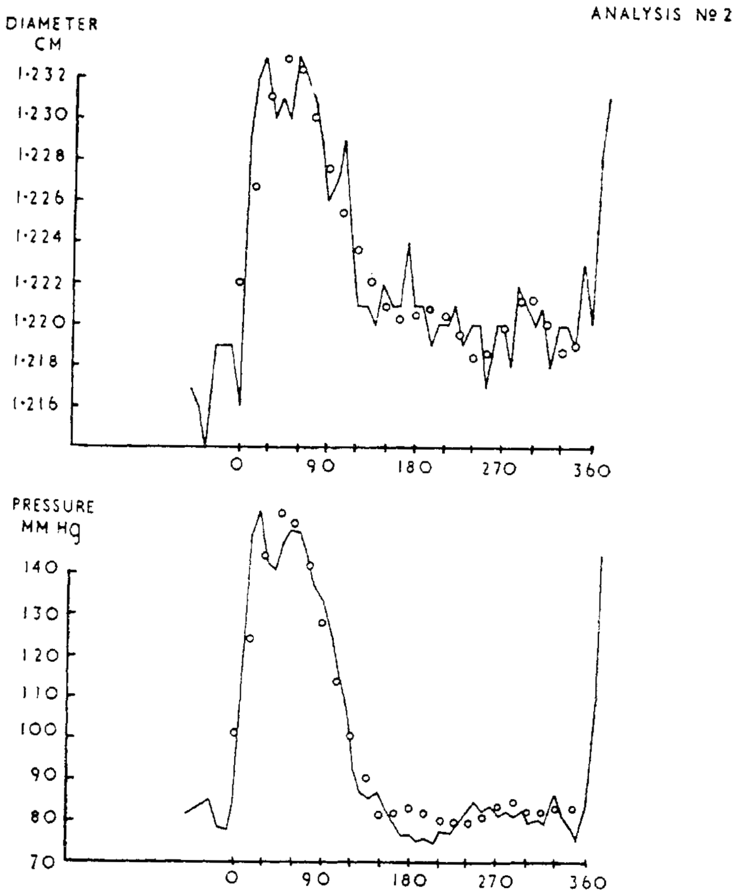

Fig. 4

T=0·352 SEC

Harmonic	Phase-lag (degrees)	
	Analysis No. 1	Analysis No. 2
1	10·6	5·5
2	4·9	−0·5
3	−4·8	15·87
4	−46·5	1·9

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SUMMARY

In the consideration of pulsatile flow in arteries, the conditions of oscillatory flow in a thin-walled elastic tube (Womersley 1955 a) have to be modified owing to the 'tethering' effect of the connective tissue and the mass of the adjacent organs. It is shown that by modification of a single parameter (the ratio of the wall-thickness to the radius of the tube) the equations of the earlier paper can be used to describe the motion when the tube is constrained as it is in the living body. If the longitudinal constraint is very stiff the solution is a simple one, for the relationship between pressure-gradient and rate of flow reduces to that of the rigid tube (Womersley 1955 b) but the wave velocity remains finite.

The application of the theory to the relation of arterial dilatation and pressure show that it predicts the data as accurately as the experimental measurements allow.

RÉSUMÉ

Afin de pouvoir examiner le courant à pulsations dans les artères, on doit modifier les conditions du flux oscillant dans un tube élastique à parois minces (Womersley 1955 a) vu l'influence 'freinante' du tissu connectif et de la masse des organes voisins. On montre qu'en modifiant un seul paramètre (le rapport entre l'épaisseur de paroi et le rayon du tube) on peut employer les équations dérivées dans un article antérieur pour décrire le mouvement, si le tube est contraint ainsi comme c'est le cas dans un organisme vivant. Si la contrainte longitudinale est très rigide, la solution est bien simple, parce que la relation entre le gradient de pression et la vélocité du courant est réduite à celle obtenue pour un tube rigide (Womersley 1955 b), tandis que la vélocité d'onde continue d'être finie.

L'application de la théorie à la relation entre la dilatation artérielle et la pression montre que la théorie prédit les résultats aussi exactement que le permettent les mesures experimentales.

ZUSAMMENFASSUNG

Wenn man die pulsierende Strömung in Arterien betrachtet, muss man die Bedingungen der oszillierenden Strömung in einer dünnwandigen elastischen Röhre (Womersley 1955 a) infolge hemmender Wirkung des Bindegewebes sowie der Masse der Nachbarorgane abändern. Es wird gezeigt, dass nach Veränderung eines einzelnen Parameters (des Verhältnisses zwischen Wandstärke und Röhrenradius) die in einem früheren Aufsatz angegebenen Gleichungen zur Beschreibung der Bewegung (falls die Röhre, wie im lebendigen Körper, verjüngt ist) benutzt werden. Falls die longitudinale Fixierung sehr steif ist, ist die Lösung der Gleichungen sehr einfach, da dann die Abhängigkeit zwischen dem Druckgradienten und der Strömungsgeschwindigkeit derjenigen in einer steifen Röhre gleich wird (Womersley 1955 b), wobei jedoch die Wellengeschwindigkeit endlich bleibt.

Die Anwendung der Theorie zur Abhängigkeit zwischen der arteriellen Ausdehnung und Druck zeigt, dass die Ergebnisse so genau von der Theorie vorausgesagt werden, wie es die Experimentalmessungen erlauben.

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