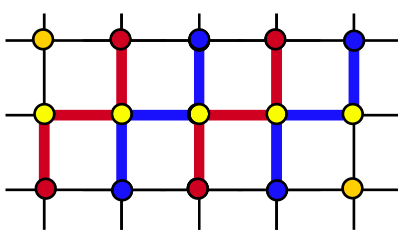
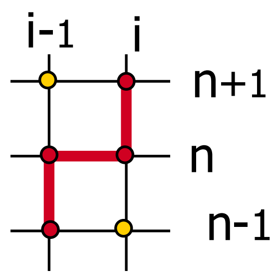
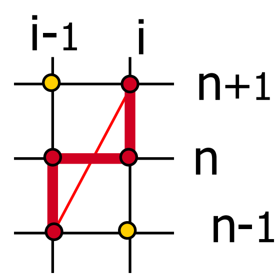
МГУ имени М.В. Ломоносова, ИБРАЭ РАН

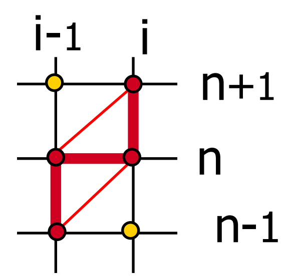
Расширенная форма законов сохранения гиперболического типа

Головизнин В.М.









ВМК, 2025

Одномерное скалярное квазилинейное уравнение гиперболического типа

$$\frac{\partial u}{\partial t}+\frac{\partial F(u)}{\partial x}=0;$$ Закон сохранения - Консервативная форма

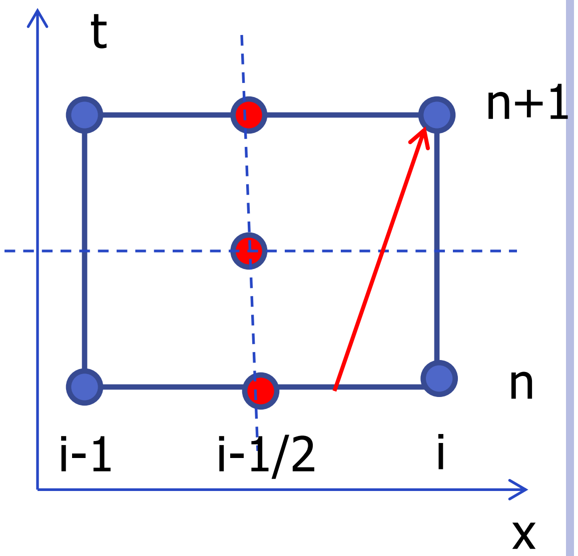
$$\frac{\partial u}{\partial t}+c(u)\frac{\partial u}{\partial x}=0; \qquad c(u)=\frac{\partial F(u)}{\partial u};$$ Характеристическая форма

Балансно – характеристические схемы второго порядка

Предиктор $$\frac{u_{i-1/2}^{n+1/2}-u_{i-1/2}^{n}}{\tau/2}+\frac{F(u_i^n)-F(u_{i-1}^n)}{x_i-x_{i-1}}=0;$$

Генератор потоков ???????

Корректор $$\frac{u_{i-1/2}^{n+1}-u_{i-1/2}^{n+1/2}}{\tau/2}+\frac{F(u_i^{n+1})-F(u_{i-1}^{n+1})}{x_i-x_{i-1}}=0;$$



$$u_{i-1/2}=\frac{1}{x_i-x_{i-1}}\int_{x_{i-1}}^{x_i}u(x)\,dx$$ Средние по ячейкам

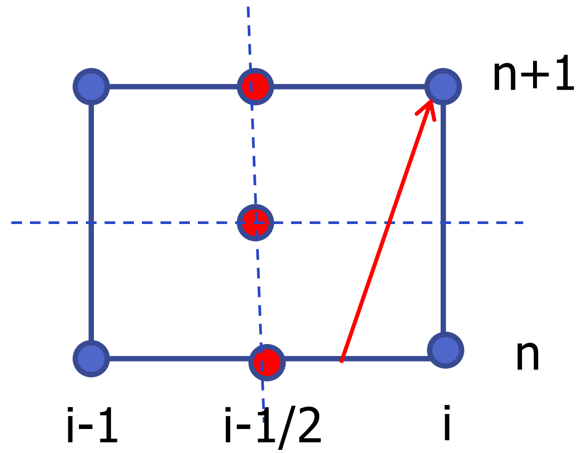
Интерполяционные балансно-характеристические схемы

Подсеточное восполнение

$$u(x) = ax^2 + bx + c, \quad x \in [x_i, x_{i-1}];$$

$$u_i = a(x_i)^2 + bx_i + c; \quad u_{i-1} = a(x_{i-1})^2 + bx_i - 1 + c;$$

$$u_{i-1/2} = \frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} \left(ax^2 + bx + c\right) dx;$$



Перенос по характеристике $\quad u_i^{n+1} = u\left(x^*\right); \quad x^* = x_i - c_{i-1/2}^{n+1/2}\tau;$

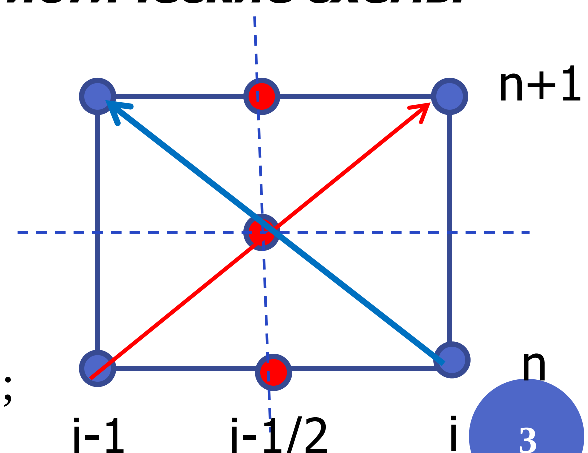
Экстраполяционные балансно-характеристические схемы

$$u_i^{n+1} = 2u_{i-1/2}^{n+1/2} - u_{i-1}^n, \quad if \quad c_{i-1/2}^{n+1/2} > 0;$$

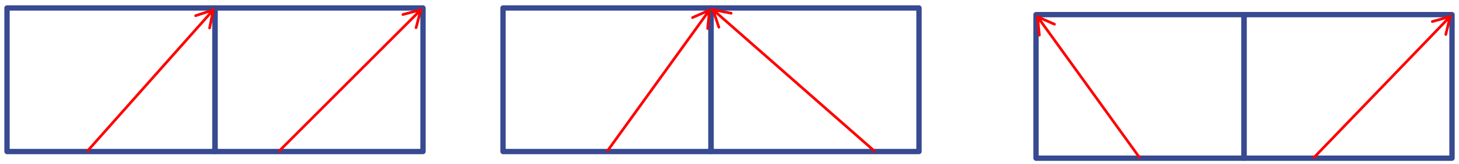
$$u_{i-1}^{n+1} = 2u_{i-1/2}^{n+1/2} - u_i^n, \quad if \quad c_{i-1/2}^{n+1/2} < 0;$$

$$\left(u_i^{n+1}, u_{i-1}^{n+1}\right) \in [u_{\max}, u_{\min}];$$

$$u_{\max} = \max\left\{u_i^n, u_{i-1/2}^n, u_{i-1}^n\right\}; \quad u_{\min} = \min\left\{u_i^n, u_{i-1/2}^n, u_{i-1}^n\right\};$$



Проблема звуковых точек в характеристических алгоритмах.



TVD-подход, минимизация локальных вариаций
(Головизнин В.М., Канаев А.А)

Переход к релаксационному приближению (The Jin-Xin model)
(Головизнин В.М., Сержантов А.В.)

$$\frac{\partial u}{\partial t}+\frac{\partial q}{\partial x}=0; \quad \frac{\partial q}{\partial t}+a^2\frac{\partial u}{\partial x}=\frac{F(u)-q}{\varepsilon}; \quad a=const;$$

Характеристическая форма в инвариантах Римана

$$\frac{\partial R}{\partial t}+a\frac{\partial R}{\partial x}=g; \quad \frac{\partial Q}{\partial t}-a\frac{\partial Q}{\partial x}=g;$$

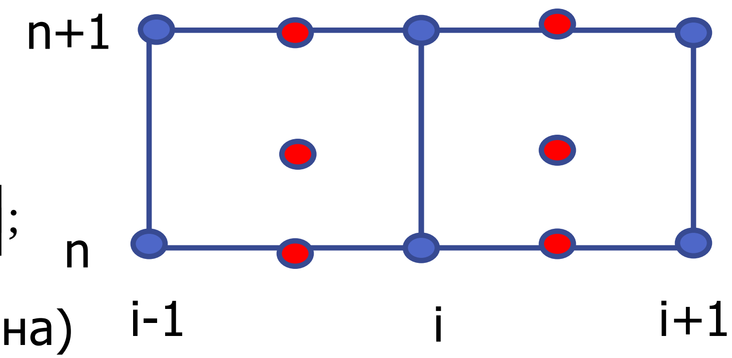
$$R=q+au; \qquad Q=q-au; \qquad g=\frac{F(u)-q}{\varepsilon};$$

Схема КАБАРЕ для релаксационного приближения

Фаза 1 (предиктор)

$$\frac{u_{i+1/2}^{n+1/2}-u_{i+1/2}^{n}}{\tau/2}+\frac{q_{i+1}^{n}-q_{i}^{n}}{\Delta x}=0;$$

$$\frac{q_{i+1/2}^{n+1/2}-q_{i+1/2}^{n}}{\tau/2}+c^2\frac{u_{i+1}^{n}-u_{i}^{n}}{\Delta x}=\frac{1}{\varepsilon}\left[F\left(u_{i+1/2}^{n+1/2}\right)-q_{i+1/2}^{n+1/2}\right];$$



Фаза 2 (экстраполяция инвариантов Римана)

$$R_i^{n+1}=2R_{i+1/2}^{n+1/2}-R_{i-1}^{n};\qquad Q_i^{n+1}=2Q_{i+1/2}^{n+1/2}-Q_{i+1}^{n};$$

Коррекция потоков

$$R_i^{n+1}=\begin{cases}R_i^{n+1} & if\quad \left(\min R_{i-1/2}\le R_i^{n+1}\le \max R_{i-1/2}\right)\\ \min R_{i-1/2} & if\quad \left(R_i^{n+1}<\min R_{i-1/2}\right)\\ \max R_{i-1/2} & if\quad \left(R_i^{n+1}>\max R_{i-1/2}\right)\end{cases}\qquad Q_i^{n+1}=\begin{cases}Q_i^{n+1} & if\quad \left(\min Q_{i+1/2}\le Q_i^{n+1}\le \max Q_{i+1/2}\right)\\ \min Q_{i+1/2} & if\quad \left(Q_i^{n+1}<\min Q_{i+1/2}\right)\\ \max Q_{i+1/2} & if\quad \left(Q_i^{n+1}>\max Q_{i+1/2}\right)\end{cases}$$

Ограничители

$$\min R_{i-1/2}=\min\left\{R_{i-1}^{n},R_{i-1/2}^{n},R_{i}^{n}\right\}+\tau\left\langle g_1\right\rangle_{i-1/2}^{n};$$

$$\max R_{i-1/2}=\max\left\{R_{i-1}^{n},R_{i-1/2}^{n},R_{i}^{n}\right\}+\tau\left\langle g_1\right\rangle_{i-1/2}^{n};$$

$$\min Q_{i+1/2}=\min\left\{Q_{i}^{n},Q_{i+1/2}^{n},Q_{i+1}^{n}\right\}+\tau\left\langle g_2\right\rangle_{i+1/2}^{n};$$

$$\max Q_{i+1/2}=\max\left\{Q_{i}^{n},Q_{i+1/2}^{n},Q_{i+1}^{n}\right\}+\tau\left\langle g_2\right\rangle_{i+1/2}^{n};$$

$$\left\langle g_1\right\rangle_{i-1/2}^{n}=\frac{R_{i-1/2}^{n+1/2}-R_{i-1/2}^{n}}{\tau/2}+c\cdot\frac{R_{i}^{n}-R_{i-1}^{n}}{\Delta x};$$

$$\left\langle g_2\right\rangle_{i+1/2}^{n}=\frac{Q_{i+1/2}^{n+1/2}-Q_{i+1/2}^{n}}{\tau/2}-c\cdot\frac{Q_{i+1}^{n}-Q_{i}^{n}}{\Delta x};$$

Новые потоковые переменные

$$q_i^{n+1} = \left(R_i^{n+1} + Q_i^{n+1}\right)\big/2; \qquad u_i^{n+1} = \left(R_i^{n+1} - Q_i^{n+1}\right)\big/2c;$$

Фаза 3 (корректор)

$$\frac{u_{i+1/2}^{n+1} - u_{i+1/2}^{n}}{\tau} + \frac{1}{2}\left(\frac{q_{i+1}^{n+1} - q_i^{n+1}}{\Delta x} + \frac{q_{i+1}^{n} - q_i^{n}}{\Delta x}\right) = 0;$$

$$\frac{q_{i+1/2}^{n+1} - q_{i+1/2}^{n}}{\tau} + \frac{c^2}{2}\left(\frac{u_{i+1}^{n+1} - u_i^{n+1}}{\Delta x} + \frac{u_{i+1}^{n} - u_i^{n}}{\Delta x}\right) =$$

$$= \frac{1}{2\varepsilon}\left\{\left[F\left(u_{i+1/2}^{n+1}\right) - q_{i+1/2}^{n+1}\right] + \left[F\left(u_{i+1/2}^{n}\right) - q_{i+1/2}^{n}\right]\right\};$$

Требование на параметр a

Цзинь-Синь (The Jin-Xin model)

$$\frac{\partial u}{\partial t}+\frac{\partial q}{\partial x}=0; \qquad \frac{\partial q}{\partial t}+a^2\frac{\partial u}{\partial x}=\frac{1}{\varepsilon}\left[F(u)-q\right]; \quad \varepsilon \text{ -}$$ малый параметр

a^2 - константа, величину которой нужно определить

Из второго уравнения системы находим $$q=F(u)-\varepsilon\left[\frac{\partial q}{\partial t}+c^2\frac{\partial u}{\partial x}\right]$$

Подставим это q в производную $$\partial q/\partial t \Rightarrow q=F(u)-\varepsilon\left[\frac{\partial F(u)}{\partial t}+c^2\frac{\partial u}{\partial x}\right]+O(\varepsilon^2);$$

$$q=F(u)-\varepsilon\left[\frac{\partial F(u)}{\partial t}+c^2\frac{\partial u}{\partial x}\right]+O(\varepsilon^2)=F(u)-\varepsilon\left[\frac{\partial F(u)}{\partial u}\frac{\partial u}{\partial t}+c^2\frac{\partial u}{\partial x}\right]+O(\varepsilon^2)=$$

$$=F(u)-\varepsilon\left[-\frac{\partial F(u)}{\partial u}\frac{\partial F(u)}{\partial x}+c^2\frac{\partial u}{\partial x}\right]+O(\varepsilon^2)=F(u)-\varepsilon\left[c^2-\frac{\partial F(u)}{\partial u}\frac{\partial F(u)}{\partial u}\right]\frac{\partial u}{\partial x}+O(\varepsilon^2)=$$

$$=F(u)-\varepsilon\left\{c^2-\left[F'(u)\right]^2\right\}\frac{\partial u}{\partial x}+O(\varepsilon^2);$$

Подставим это значение потока в первое уравнение системы

$$\frac{\partial u}{\partial t}+\frac{\partial F\left(u\right)}{\partial x}=\varepsilon\frac{\partial}{\partial x}\left\{c^{2}-\left[F'\left(u\right)\right]^{2}\right\}\frac{\partial u}{\partial x}+O\left(\varepsilon^{2}\right);$$

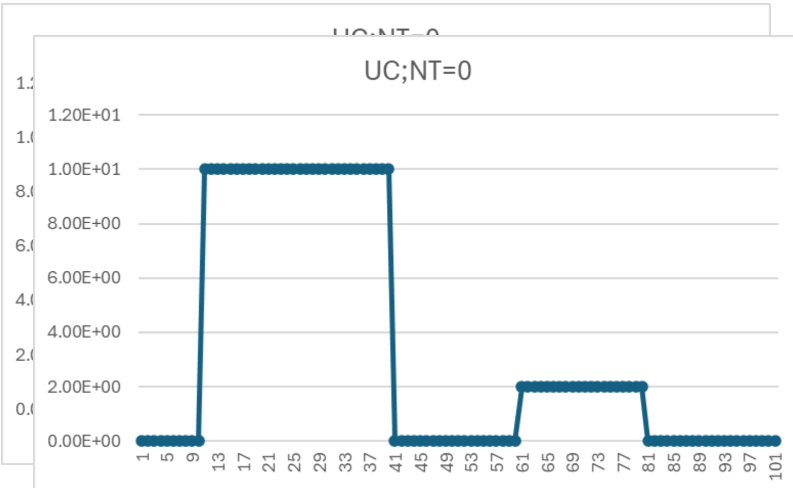
Для того, чтобы решение было ограниченным, необходимо, чтобы

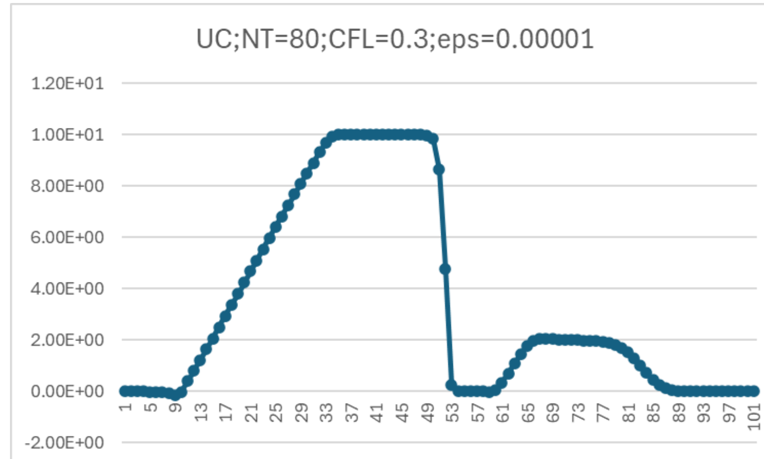
выполнялось неравенство $c^{2}-\left[F'\left(u\right)\right]^{2}>0;$

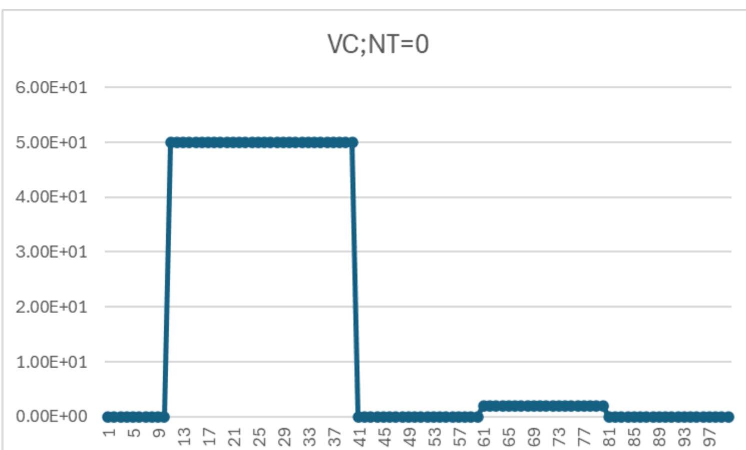
Тестовые расчеты

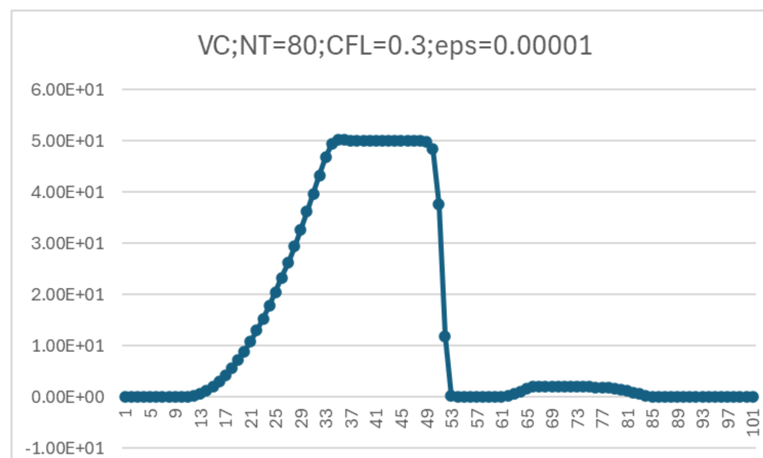
Тестовые расчеты. Варьирование ε

$$c^2 = \max_{i,n=1} \left|F'(u)\right|^2 \qquad F(u) = \frac{u^2}{2}$$



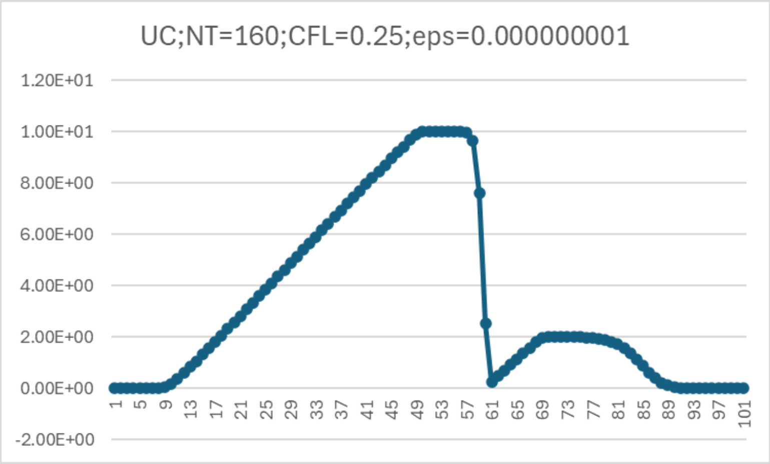


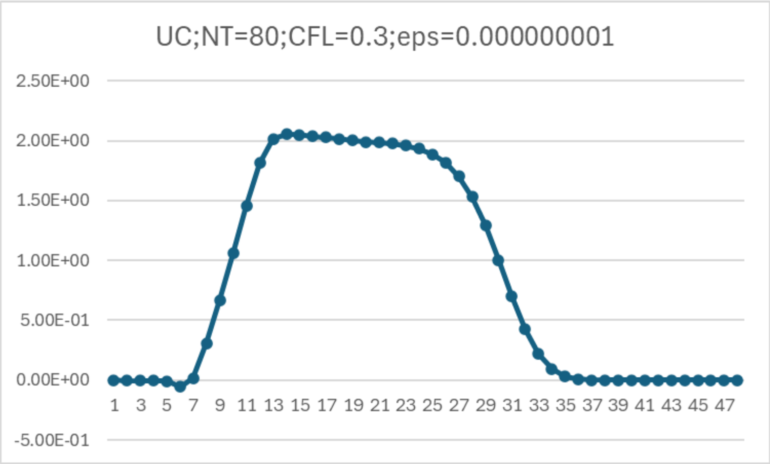


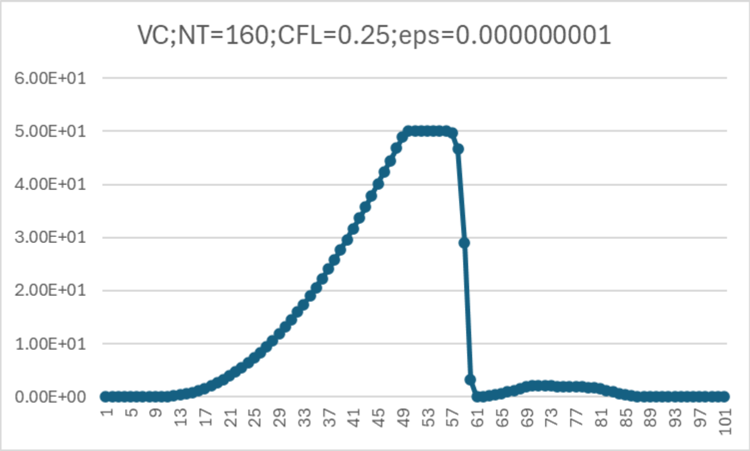


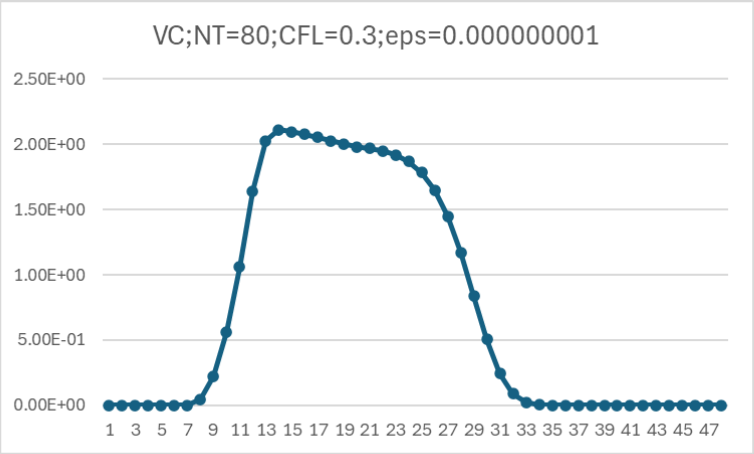
Тестовые расчеты.

$$c^2 = \max_{i,n=1} \left| F'(u) \right|^2 \qquad F(u) = \frac{u^2}{2}$$









Нелинейная релаксационная модель

$$\frac{\partial u}{\partial t}+\frac{\partial q}{\partial x}=0; \qquad \frac{\partial q}{\partial t}+c^2(u)\cdot\frac{\partial u}{\partial x}=g=\frac{1}{\varepsilon}\left[F(u)-q\right]; \quad c(u)=\left|F'(u)\right| ;$$

Характеристическая форма

$$\left(\frac{\partial q}{\partial t}+c(u)\frac{\partial u}{\partial t}\right)+c(u)\left(\frac{\partial q}{\partial x}+c(u)\frac{\partial u}{\partial x}\right)=g;$$

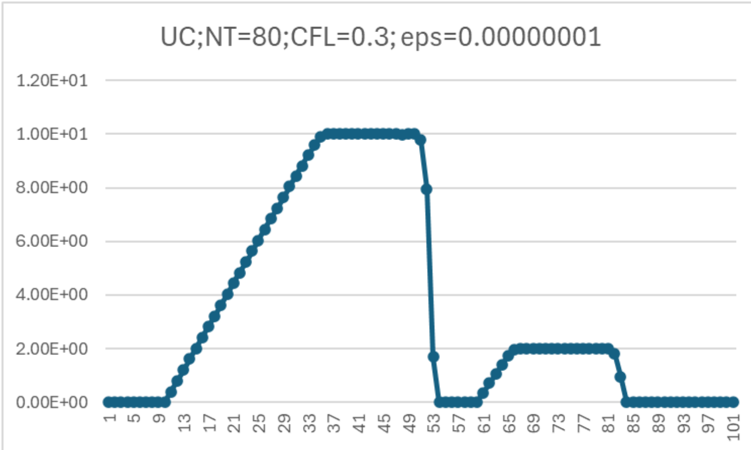
$$\left(\frac{\partial q}{\partial t}-c(u)\frac{\partial u}{\partial t}\right)-c(u)\left(\frac{\partial q}{\partial x}-c(u)\frac{\partial u}{\partial x}\right)=g;$$

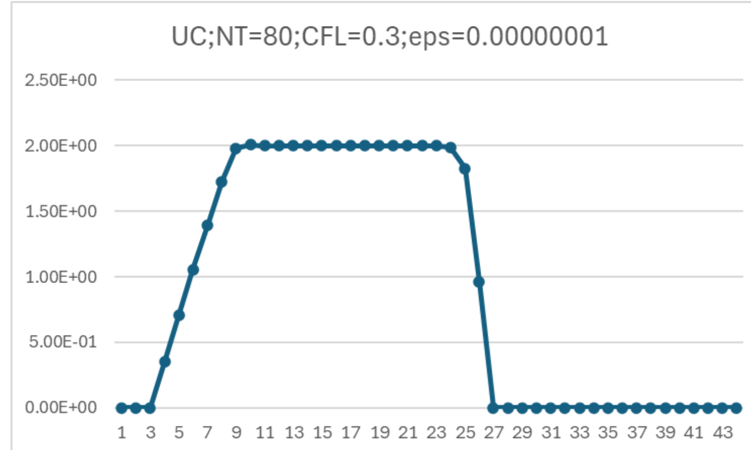
Линейные инварианты

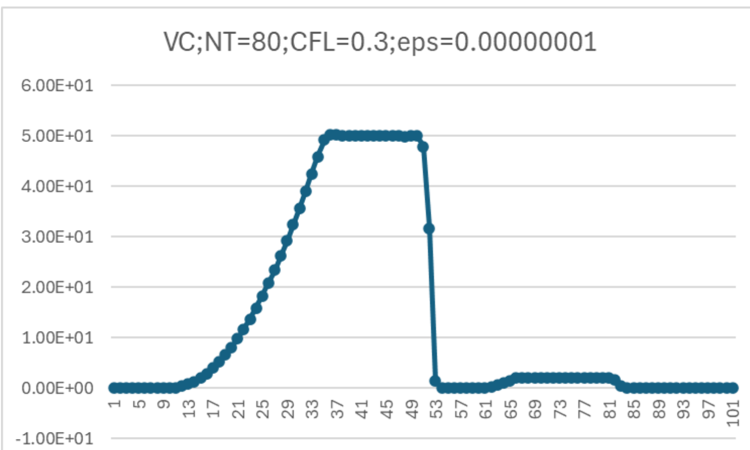
$$R_{i-1/2}=q+c_{i-1/2}\cdot u; \qquad Q_{i-1/2}=q-c_{i-1/2}\cdot u;$$

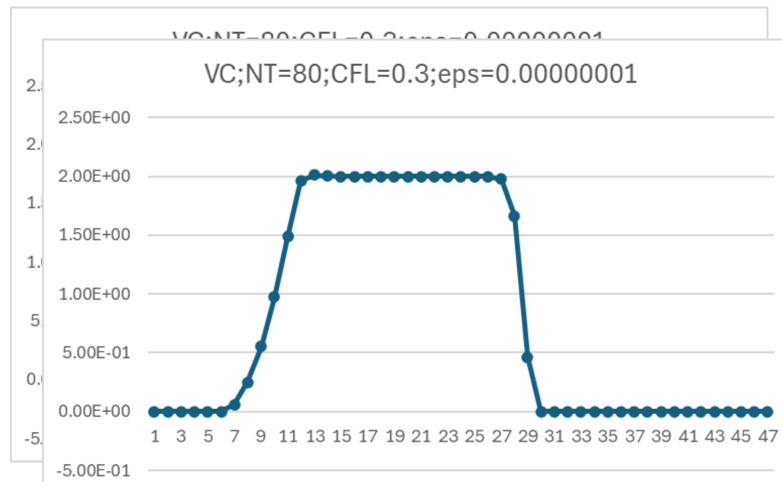
$$c_{i-1/2}=1.1\cdot\max\left\{\sqrt{\left|F'\right|_i^n};\sqrt{\left|F'\right|_{i-1/2}^n};\sqrt{\left|F'\right|_{i-3/2}^n}\right\}+\varepsilon_2;$$

$$c_{i-1/2}=1.1\cdot\max\left\{\sqrt{\left|F'\right|_{i}^{n}};\sqrt{\left|F'\right|_{i-1/2}^{n}};\sqrt{\left|F'\right|_{i-3/2}^{n}}\right\}+\varepsilon_2; \qquad F(u)=\frac{u^2}{2}$$







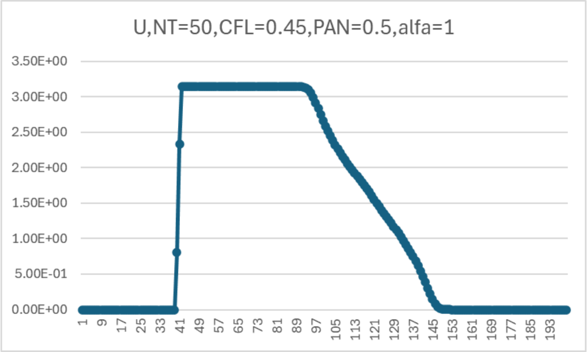


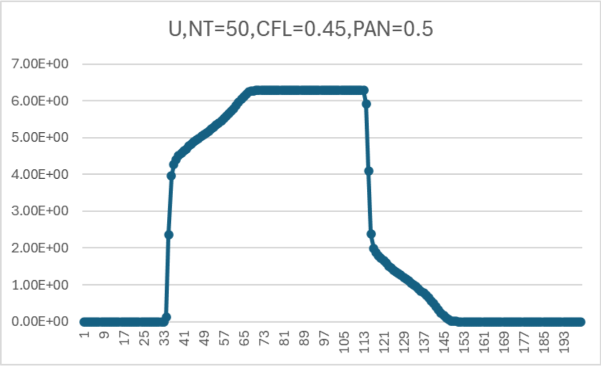
Задание. Освоить фортранную программу и Exel,
 Провести серии расчетов с другими потоковыми функциями.
Написать отчет и сделать презентацию. Провести сравнение с аналитикой

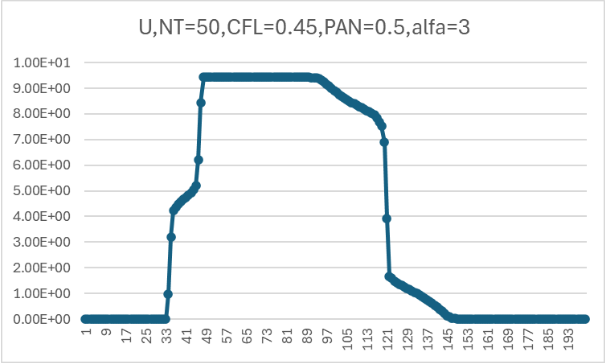
Задача №5. $f\left(u\right)=\sin(u)$ $x\in\left[0,1\right], N_x=201, CFL=0.25, PAN=0.5$

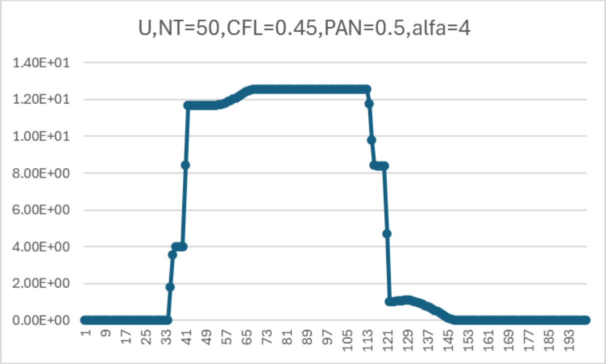
,

$$u(x,0)=\begin{cases}0, & x<0.2\\ \alpha\cdot\pi, & 0.2\le x\le 0.6;\\ 0, & 0.6<x\le 1\end{cases}$$









Релаксационное приближение гиперболических систем.

S. Jin and Z. P. Xin. The relaxation schemes for systems of conservation laws in arbitrary space dimensions. Comm. Pure Appl. Math., 48(3):235–276, 1995.

На примере мелкой воды

$$\frac{\partial H}{\partial t}+\frac{\partial Hu}{\partial x}=0; \qquad \frac{\partial Hu}{\partial t}+\frac{\partial\left(Hu^2+0.5gH^2\right)}{\partial x}=0;$$

Введем обозначения

$$u_1=H; \quad u_2=Hu; \quad v_1=f_1=Hu; \quad v_2=f_2=Hu^2+0.5g\cdot H^2;$$

Релаксационное приближение

$$\frac{\partial u_1}{\partial t}+\frac{\partial v_1}{\partial x}=0; \qquad \frac{\partial u_2}{\partial t}+\frac{\partial v_2}{\partial x}=0;$$

$$\frac{\partial v_1}{\partial t}+a^2\cdot\frac{\partial u_1}{\partial x}=\frac{f_1-v_1}{\varepsilon}; \qquad \frac{\partial v_2}{\partial t}+a^2\cdot\frac{\partial u_2}{\partial x}=\frac{f_2-v_2}{\varepsilon};$$

Разложение Чепмена -Энскога

$$v_1 = f_1 - \varepsilon\left(\frac{\partial q_1}{\partial t} + a^2 \cdot \frac{\partial u_1}{\partial x}\right); \qquad v_2 = f_2 - \varepsilon\left(\frac{\partial v_2}{\partial t} + a^2 \cdot \frac{\partial u_2}{\partial x}\right);$$

$$\frac{\partial u_1}{\partial t} + \frac{\partial f_1}{\partial x} = \varepsilon \frac{\partial}{\partial x}\left(\frac{\partial v_1}{\partial t} + a^2 \cdot \frac{\partial u_1}{\partial x}\right); \qquad \frac{\partial u_2}{\partial t} + \frac{\partial f_2}{\partial x} = \varepsilon \frac{\partial}{\partial x}\left(\frac{\partial v_2}{\partial t} + a^2 \cdot \frac{\partial u_2}{\partial x}\right);$$

$$\frac{\partial v_1}{\partial t} = \frac{\partial v_1}{\partial u_1}\frac{\partial u_1}{\partial t} + \frac{\partial v_1}{\partial u_2}\frac{\partial u_2}{\partial t}; \qquad \frac{\partial v_2}{\partial t} = \frac{\partial v_2}{\partial u_1}\frac{\partial u_1}{\partial t} + \frac{\partial v_2}{\partial u_2}\frac{\partial u_2}{\partial t};$$

Получаем $$\frac{\partial \vec{\psi}}{\partial t} + \frac{\partial \vec{F}}{\partial x} = \varepsilon \frac{\partial}{\partial x}\mathbf{D}\frac{\partial \vec{\psi}}{\partial x}; \qquad \vec{\psi} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}; \qquad \vec{F} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix};$$

$$\mathbf{D} = \begin{bmatrix} a^2 - \left(\frac{\partial f_1}{\partial u_1}\frac{\partial f_1}{\partial u_1} + \frac{\partial f_1}{\partial u_2}\frac{\partial f_2}{\partial u_1}\right) & -\left(\frac{\partial f_1}{\partial u_1}\frac{\partial f_1}{\partial u_2} + \frac{\partial f_1}{\partial u_2}\frac{\partial f_2}{\partial u_2}\right) \\ -\left(\frac{\partial f_2}{\partial u_1}\frac{\partial f_1}{\partial u_1} + \frac{\partial f_2}{\partial u_2}\frac{\partial f_2}{\partial u_1}\right) & a^2 - \left(\frac{\partial f_2}{\partial u_1}\frac{\partial f_1}{\partial u_2} + \frac{\partial f_2}{\partial u_2}\frac{\partial f_2}{\partial u_2}\right) \end{bmatrix};$$

Характеристическая форма релаксационного приближения

$$\left(\frac{\partial v_1}{\partial t} + |a| \cdot \frac{\partial u_1}{\partial t}\right) + |a|\left(\frac{\partial v_1}{\partial x} + |a| \cdot \frac{\partial u_1}{\partial x}\right) = \frac{f_1 - v_1}{\varepsilon}; \quad \left(\frac{\partial v_2}{\partial t} + |a| \cdot \frac{\partial u_2}{\partial t}\right) + |a|\left(\frac{\partial v_2}{\partial x} + |a| \cdot \frac{\partial u_2}{\partial x}\right) = \frac{f_2 - v_2}{\varepsilon};$$

$$\left(\frac{\partial v_1}{\partial t} - |a| \cdot \frac{\partial u_1}{\partial t}\right) - |a|\left(\frac{\partial v_1}{\partial x} - |a| \cdot \frac{\partial u_1}{\partial x}\right) = \frac{f_1 - v_1}{\varepsilon}; \quad \left(\frac{\partial v_2}{\partial t} - |a| \cdot \frac{\partial u_2}{\partial t}\right) - |a|\left(\frac{\partial v_2}{\partial x} - |a| \cdot \frac{\partial u_2}{\partial x}\right) = \frac{f_2 - v_2}{\varepsilon}$$

Особенность реализации схемы КАБАРЕ

Фаза 1

$$\frac{(u_1)_c^{n+1/2}-(u_1)_c^n}{\tau/2}+\frac{(v_1)_R^n-(v_1)_L^n}{\Delta x}=0; \quad \frac{(u_2)_c^{n+1/2}-(u_2)_c^n}{\tau/2}+\frac{(v_2)_R^n-(v_2)_L^n}{\Delta x}=0;$$

$$\frac{(v_1)_c^{n+1/2}-(v_1)_c^n}{\tau/2}+c^2\frac{(u_1)_R^n-(u_1)_L^n}{\Delta x}=\frac{f_1\left[(u_1)_c^{n+1/2},(u_2)_c^{n+1/2}\right]-(v_1)_c^{n+1/2}}{\varepsilon};$$

$$\frac{(v_2)_c^{n+1/2}-(v_2)_c^n}{\tau/2}+c^2\frac{(u_2)_R^n-(u_2)_L^n}{\Delta x}=\frac{f_2\left[(u_1)_c^{n+1/2},(u_2)_c^{n+1/2}\right]-(v_2)_c^{n+1/2}}{\varepsilon};$$

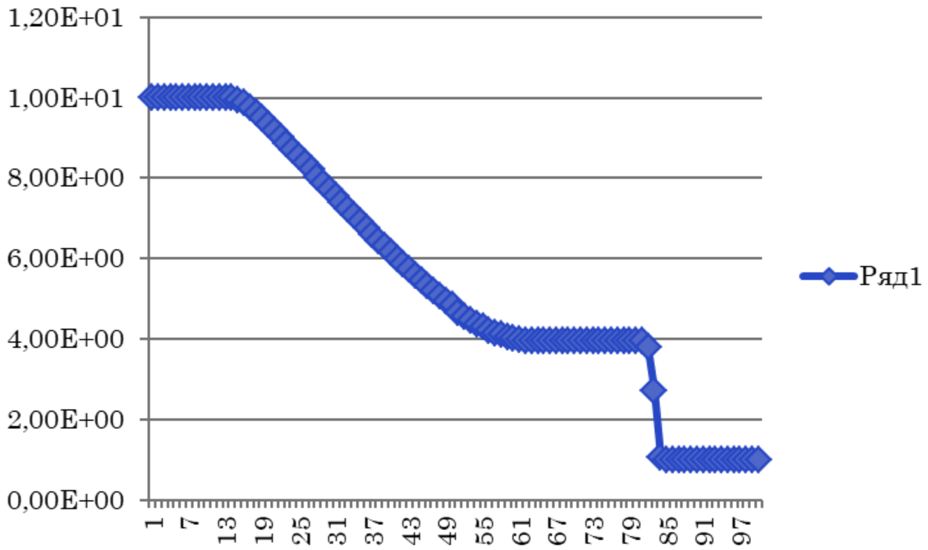
Фаза 3

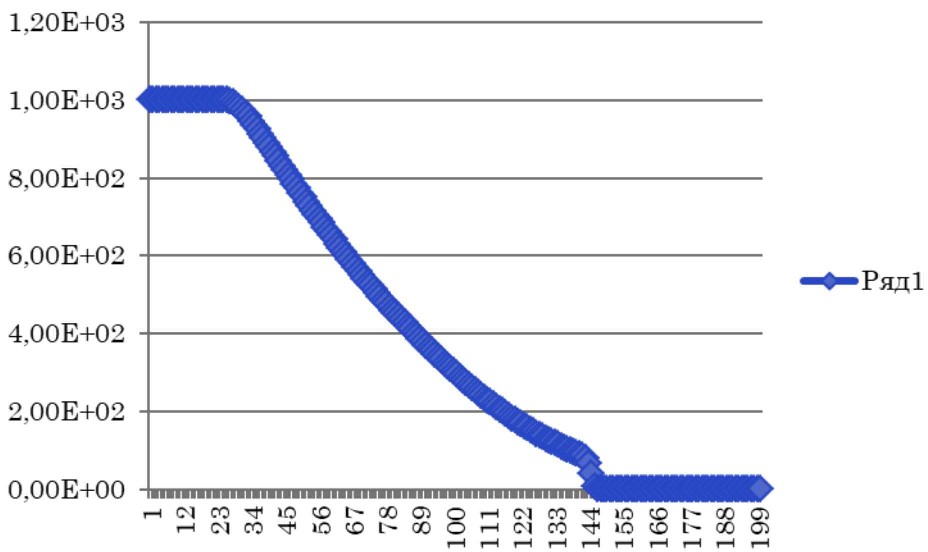
$$\frac{(u_1)_c^{n+1}-(u_1)_c^{n+1/2}}{\tau/2}+\frac{(v_1)_R^{n+1}-(v_1)_L^{n+1}}{\Delta x}=0; \quad \frac{(u_2)_c^{n+1}-(u_2)_c^{n+1/2}}{\tau/2}+\frac{(v_2)_R^{n+1}-(v_2)_L^{n+1}}{\Delta x}=0;$$

$$\frac{(v_1)_c^{n+1}-(v_1)_c^{n+1/2}}{\tau/2}+c^2\frac{(u_1)_R^{n+1}-(u_1)_L^{n+1}}{\Delta x}=\frac{f_1\left[(u_1)_c^{n+1},(u_2)_c^{n+1}\right]-(v_1)_c^{n+1}}{\varepsilon};$$

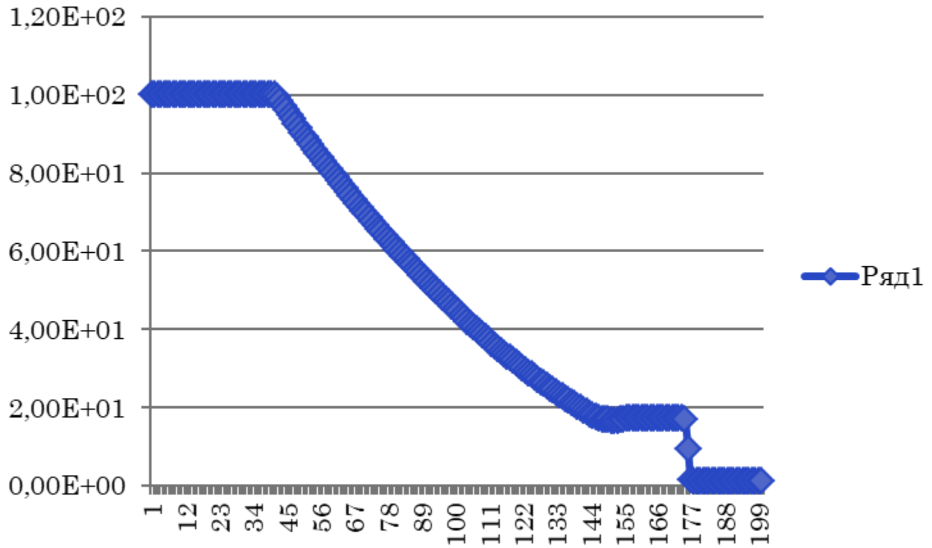
$$\frac{(v_2)_c^{n+1}-(v_2)_c^{n+1/2}}{\tau/2}+c^2\frac{(u_2)_R^{n+1}-(u_2)_L^{n+1}}{\Delta x}=\frac{f_2\left[(u_1)_c^{n+1},(u_2)_c^{n+1}\right]-(v_2)_c^{n+1}}{\varepsilon};$$

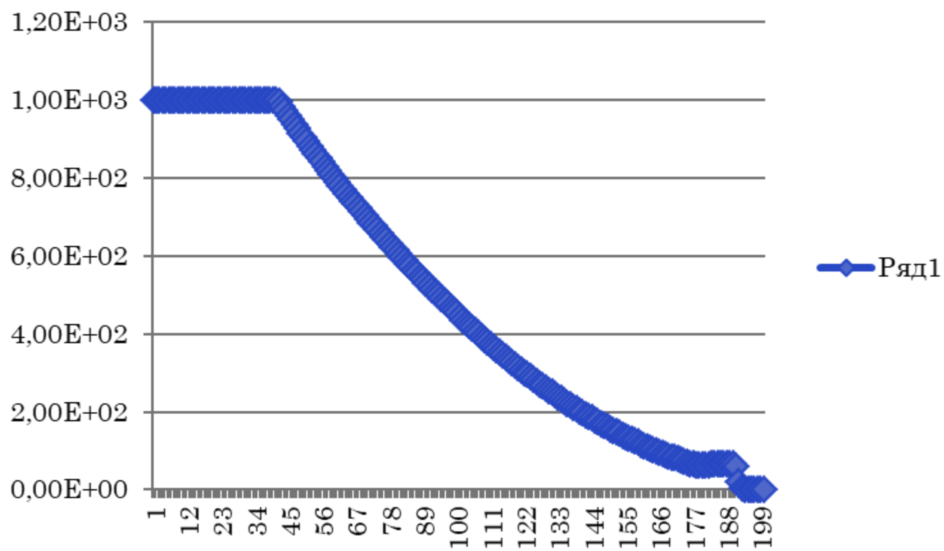
Схема КАБАРЕ для релаксационного приближения





Расчеты по схеме КАБАРЕ исходной системы с обработкой звуковых точек по релаксационному алгоритму





Отказ от релаксационной модели

$$\frac{\partial u}{\partial t}+\frac{\partial F(u)}{\partial x}=0;$$ Консервативная форма уравнения

Представление в виде системы дифференциально – алгебраических уравнений

$$\frac{\partial u}{\partial t}+\frac{\partial v}{\partial x}=0; \qquad v=F(u);$$

Замена алгебраического уравнения эволюционным

$$\frac{\partial v}{\partial t}=\frac{\partial v}{\partial u}\frac{\partial u}{\partial t}=-\frac{\partial v}{\partial u}\frac{\partial v}{\partial x}=-\left(\frac{\partial v}{\partial u}\right)^2\frac{\partial u}{\partial x}=-c^2(u)\cdot\frac{\partial u}{\partial x}; \qquad c(u)=\left|\frac{\partial F(u)}{\partial u}\right|;$$

Система из двух дифференциальных уравнений

$$\frac{\partial u}{\partial t}+\frac{\partial v}{\partial x}=0; \qquad \frac{\partial v}{\partial t}+c^2(u)\cdot\frac{\partial u}{\partial x}=0;$$

Система характеристических уравнений

$$\left(\frac{\partial q}{\partial t}+c(u)\frac{\partial u}{\partial t}\right)+c(u)\left(\frac{\partial q}{\partial x}+c(u)\frac{\partial u}{\partial x}\right)=0;$$

$$\left(\frac{\partial q}{\partial t}-c(u)\frac{\partial u}{\partial t}\right)-c(u)\left(\frac{\partial q}{\partial x}-c(u)\frac{\partial u}{\partial x}\right)=0;$$

Расширенная форма уравнений мелкой воды

Исходная система уравнений. $\frac{\partial h}{\partial t}+\frac{\partial hu}{\partial x}=0; \qquad \frac{\partial hu}{\partial t}+\frac{\partial}{\partial x}\left(hu^2+0.5\cdot g\cdot h^2\right)=0;$

Расширенная дифференциально – алгебраическая форма.

$$\frac{\partial u_1}{\partial t}+\frac{\partial v_1}{\partial x}=0; \qquad \frac{\partial u_2}{\partial t}+\frac{\partial v_2}{\partial x}=0; \qquad \left(u_1=h; \qquad u_2=hu;\right)$$

$$v_1=u_2; \qquad\qquad v_2=\frac{u_2^2}{u_1}+\frac{g\cdot u_1^2}{2};$$

Заменим первое алгебраические соотношение эволюционным уравнением.

$$\frac{\partial v_1}{\partial t}=\frac{\partial v_1}{\partial u_2}\frac{\partial u_2}{\partial t}=-\frac{\partial v_2}{\partial x}=-\left(\frac{\partial v_2}{\partial u_1}\frac{\partial u_1}{\partial x}+\frac{\partial v_2}{\partial u_2}\frac{\partial u_2}{\partial x}\right)=-\left[\left(-u^2+c^2\right)\frac{\partial u_1}{\partial x}+2u\frac{\partial u_2}{\partial x}\right]$$

В результате получаем

$$\frac{\partial v_1}{\partial t}+\left(c^2-u^2\right)\frac{\partial u_1}{\partial x}+2u\frac{\partial u_2}{\partial x}=0; \qquad\qquad c^2=gH;$$

Аналогичная замена второго алгебраического уравнения на эволюционное приводит к расширенной системе дифференциальных уравнений мелкой воды.

$$\boxed{\begin{aligned}&\frac{\partial u_1}{\partial t}+\frac{\partial v_1}{\partial x}=0;\\&\frac{\partial v_1}{\partial t}+\left(gH-u^2\right)\frac{\partial u_1}{\partial x}+2u\frac{\partial u_2}{\partial x}=0;\end{aligned}}\qquad\boxed{\begin{aligned}&\frac{\partial u_2}{\partial t}+\frac{\partial v_2}{\partial x}=0;\\&\frac{\partial v_2}{\partial t}+2u\left(gH-u^2\right)\frac{\partial u_1}{\partial x}+\left(gH+3u^2\right)\frac{\partial u_2}{\partial x}=0;\end{aligned}}$$

Векторно- матричное представление.

$$\vec{\varphi}=\begin{pmatrix}u_1\\v_1\\u_2\\v_2\end{pmatrix};\quad A=\begin{pmatrix}0&1&0&0\\\left(gH-u^2\right)&0&2u&0\\0&0&0&1\\2u\left(gH-u^2\right)&0&\left(gH+3u^2\right)&0\end{pmatrix};\quad\frac{\partial\vec{\varphi}}{\partial t}+A\frac{\partial\vec{\varphi}}{\partial x}=0;$$

Собственные числа матрицы А.

$$\left\|A-\lambda E\right\|=0;\quad\lambda_1=(u+c);\quad\lambda_2=-(u+c);\quad\lambda_3=(u-c);\quad\lambda_4=-(u-c);$$

Собственные числа действительны и различны. Расширенная форма уравнений мелкой воды остается гиперболической

Характеристическая форма расширенной системы уравнений мелкой воды.

Левые собственные векторы матрицы A.

$$\vec{l}_1 = \left(\left(u^2 - c^2\right) \quad (u-c) \quad -(u+c) \quad -1\right); \qquad \vec{l}_2 = \left(-\left(u^2 - c^2\right) \quad (u-c) \quad (u+c) \quad -1\right);$$

$$\vec{l}_3 = \left(\left(u^2 - c^2\right) \quad (u+c) \quad -(u-c) \quad -1\right); \qquad \vec{l}_4 = \left(-\left(u^2 - c^2\right) \quad (u+c) \quad (u-c) \quad -1\right);$$

Характеристическая система уравнений

$$\left[\left(u^2-c^2\right)\frac{\partial u_1}{\partial t}+(u-c)\frac{\partial v_1}{\partial t}-(u+c)\frac{\partial u_2}{\partial t}-\frac{\partial v_2}{\partial t}\right]+(u+c)\left[\left(u^2-c^2\right)\frac{\partial u_1}{\partial x}+(u-c)\frac{\partial v_1}{\partial x}-(u+c)\frac{\partial u_2}{\partial x}-\frac{\partial v_2}{\partial x}\right]=0;$$

$$\left[-\left(u^2-c^2\right)\frac{\partial u_1}{\partial t}+(u-c)\frac{\partial v_1}{\partial t}+(u+c)\frac{\partial u_2}{\partial t}-\frac{\partial v_2}{\partial t}\right]-(u+c)\left[-\left(u^2-c^2\right)\frac{\partial u_1}{\partial x}+(u-c)\frac{\partial v_1}{\partial x}+(u+c)\frac{\partial u_2}{\partial x}-\frac{\partial v_2}{\partial x}\right]=0;$$

$$\left[\left(u^2-c^2\right)\frac{\partial u_1}{\partial t}+(u+c)\frac{\partial v_1}{\partial t}-(u-c)\frac{\partial u_2}{\partial t}-\frac{\partial v_2}{\partial t}\right]+(u-c)\left[\left(u^2-c^2\right)\frac{\partial u_1}{\partial x}+(u+c)\frac{\partial v_1}{\partial x}-(u-c)\frac{\partial u_2}{\partial x}-\frac{\partial v_2}{\partial x}\right]=0;$$

$$\left[-\left(u^2-c^2\right)\frac{\partial u_1}{\partial t}+(u+c)\frac{\partial v_1}{\partial t}+(u-c)\frac{\partial u_2}{\partial t}-\frac{\partial v_2}{\partial t}\right]-(u-c)\left[-\left(u^2-c^2\right)\frac{\partial u_1}{\partial x}+(u+c)\frac{\partial v_1}{\partial x}+(u-c)\frac{\partial u_2}{\partial x}-\frac{\partial v_2}{\partial x}\right]=0;$$

Расширенные системы из трех эволюционных уравнений для приближения мелкой воды.

Если вместо эволюционного уравнения на поток импульса использовать его алгебраическую форму

$$\boxed{\begin{aligned}&\frac{\partial u_1}{\partial t}+\frac{\partial v_1}{\partial x}=0;\\&\frac{\partial v_1}{\partial t}+\left(gH-u^2\right)\frac{\partial u_1}{\partial x}+2u\frac{\partial u_2}{\partial x}=0;\end{aligned}}\qquad\boxed{\begin{aligned}&\frac{\partial u_2}{\partial t}+\frac{\partial v_2}{\partial x}=0;\\&v_2=u_2^2/u_1+0.5g\cdot u_1^2;\end{aligned}}$$

то такая система теряет свойство гиперболичности.

Расширенная система из трех эволюционных уравнений с эволюционным уравнение на поток импульса

$$\boxed{\begin{aligned}&\frac{\partial u_1}{\partial t}+\frac{\partial v_1}{\partial x}=0;\\&v_1=u_2;\end{aligned}}\qquad\boxed{\begin{aligned}&\frac{\partial u_2}{\partial t}+\frac{\partial v_2}{\partial x}=0;\\&\frac{\partial v_2}{\partial t}+2u\left(gH-u^2\right)\frac{\partial u_1}{\partial x}+\left(gH+3u^2\right)\frac{\partial u_2}{\partial x}=0;\end{aligned}}$$

остается гиперболической

$$\left\| A-\lambda E \right\| = \lambda^3 - \lambda\left(c^2+3u^2\right) - 2uc^2 + 2u^3 = 0;$$

$$\lambda^3 + p\lambda + q = 0;$$

$$p = -\left(c^2+3u^2\right); q = -2u\left(c^2-u^2\right);$$

Все три корня действительны и различны

$$\varphi = arctg\left\{\frac{1}{\sqrt{3}}\cdot\frac{1}{3}c\left(c^2-9u^2\right)\frac{1}{u\left(c^2-u^2\right)}\right\}$$

$$\boxed{\begin{aligned}
&\lambda_1 = \alpha+\beta = 2r^{1/6}\cos\left(\varphi/3\right) > 0, \qquad \varphi\in\left[0,\pi/2\right]\\
&\lambda_2 = -\frac{\alpha+\beta}{2} + i\frac{\sqrt{3}}{2}\left(\alpha-\beta\right) = -r^{1/6}\left[\cos\left(\varphi/3\right) - \frac{\sqrt{3}}{2}\sin\left(\varphi/3\right)\right]\\
&\lambda_3 = -\frac{\alpha+\beta}{2} - i\frac{\sqrt{3}}{2}\left(\alpha-\beta\right) = -r^{1/6}\left[\cos\left(\varphi/3\right) + \frac{\sqrt{3}}{2}\sin\left(\varphi/3\right)\right] < 0
\end{aligned}};$$

Левые собственные векторы

$$\vec{l}_1 = \left\{ \left[\lambda_1^2 - \left(gH + 3u^2 \right) \right], \lambda_1, 1 \right\}; \vec{l}_3 = \left\{ \left[\lambda_3^2 - \left(gH + 3u^2 \right) \right], \lambda_3, 1 \right\}; \vec{l}_2 = \left\{ \left[\lambda_2^2 - \left(gH + 3u^2 \right) \right], \lambda_2, 1 \right\};$$

Характеристическая форма

$$\left\{ \left[\lambda_1^2 - \left(gH + 3u^2 \right) \right] \frac{\partial u_1}{\partial t} + \lambda_1 \frac{\partial u_2}{\partial t} + \frac{\partial v_2}{\partial t} \right\} + \lambda_1 \left\{ \left[\lambda_1^2 - \left(gH + 3u^2 \right) \right] \frac{\partial u_1}{\partial x} + \lambda_1 \frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial x} \right\} = 0;$$

$$\left\{ \left[\lambda_2^2 - \left(gH + 3u^2 \right) \right] \frac{\partial u_1}{\partial t} + \lambda_2 \frac{\partial u_2}{\partial t} + \frac{\partial v_2}{\partial t} \right\} + \lambda_2 \left\{ \left[\lambda_2^2 - \left(gH + 3u^2 \right) \right] \frac{\partial u_1}{\partial x} + \lambda_2 \frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial x} \right\} = 0;$$

$$\left\{ \left[\lambda_3^2 - \left(gH + 3u^2 \right) \right] \frac{\partial u_1}{\partial t} + \lambda_3 \frac{\partial u_2}{\partial t} + \frac{\partial v_2}{\partial t} \right\} + \lambda_3 \left\{ \left[\lambda_3^2 - \left(gH + 3u^2 \right) \right] \frac{\partial u_1}{\partial x} + \lambda_3 \frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial x} \right\} = 0;$$

Характеристические формы уравнений мелкой воды.

$$\left(\frac{\partial u}{\partial t}+\frac{g}{c}\frac{\partial H}{\partial t}\right)+(u+c)\left(\frac{\partial u}{\partial x}+\frac{g}{c}\frac{\partial H}{\partial x}\right)=0;$$

$$\left(\frac{\partial u}{\partial t}-\frac{g}{c}\frac{\partial H}{\partial t}\right)+(u-c)\left(\frac{\partial u}{\partial x}-\frac{g}{c}\frac{\partial H}{\partial x}\right)=0;$$

$$\left[(u^2-c^2)\frac{\partial u_1}{\partial t}+(u-c)\frac{\partial v_1}{\partial t}-(u+c)\frac{\partial u_2}{\partial t}-\frac{\partial v_2}{\partial t}\right]+(u+c)\left[(u^2-c^2)\frac{\partial u_1}{\partial x}+(u-c)\frac{\partial v_1}{\partial x}-(u+c)\frac{\partial u_2}{\partial x}-\frac{\partial v_2}{\partial x}\right]=0;$$

$$\left[-(u^2-c^2)\frac{\partial u_1}{\partial t}+(u-c)\frac{\partial v_1}{\partial t}+(u+c)\frac{\partial u_2}{\partial t}-\frac{\partial v_2}{\partial t}\right]-(u+c)\left[-(u^2-c^2)\frac{\partial u_1}{\partial x}+(u-c)\frac{\partial v_1}{\partial x}+(u+c)\frac{\partial u_2}{\partial x}-\frac{\partial v_2}{\partial x}\right]=0;$$

$$\left[(u^2-c^2)\frac{\partial u_1}{\partial t}+(u+c)\frac{\partial v_1}{\partial t}-(u-c)\frac{\partial u_2}{\partial t}-\frac{\partial v_2}{\partial t}\right]+(u-c)\left[(u^2-c^2)\frac{\partial u_1}{\partial x}+(u+c)\frac{\partial v_1}{\partial x}-(u-c)\frac{\partial u_2}{\partial x}-\frac{\partial v_2}{\partial x}\right]=0;$$

$$\left[-(u^2-c^2)\frac{\partial u_1}{\partial t}+(u+c)\frac{\partial v_1}{\partial t}+(u-c)\frac{\partial u_2}{\partial t}-\frac{\partial v_2}{\partial t}\right]-(u-c)\left[-(u^2-c^2)\frac{\partial u_1}{\partial x}+(u+c)\frac{\partial v_1}{\partial x}+(u-c)\frac{\partial u_2}{\partial x}-\frac{\partial v_2}{\partial x}\right]=0;$$

$$\left\{\left[\lambda_1^2-\left(gH+3u^2\right)\right]\frac{\partial u_1}{\partial t}+\lambda_1\frac{\partial u_2}{\partial t}+\frac{\partial v_2}{\partial t}\right\}+\lambda_1\left\{\left[\lambda_1^2-\left(gH+3u^2\right)\right]\frac{\partial u_1}{\partial x}+\lambda_1\frac{\partial u_2}{\partial x}+\frac{\partial v_2}{\partial x}\right\}=0;$$

$$\left\{\left[\lambda_2^2-\left(gH+3u^2\right)\right]\frac{\partial u_1}{\partial t}+\lambda_2\frac{\partial u_2}{\partial t}+\frac{\partial v_2}{\partial t}\right\}+\lambda_2\left\{\left[\lambda_2^2-\left(gH+3u^2\right)\right]\frac{\partial u_1}{\partial x}+\lambda_2\frac{\partial u_2}{\partial x}+\frac{\partial v_2}{\partial x}\right\}=0;$$

$$\left\{\left[\lambda_3^2-\left(gH+3u^2\right)\right]\frac{\partial u_1}{\partial t}+\lambda_3\frac{\partial u_2}{\partial t}+\frac{\partial v_2}{\partial t}\right\}+\lambda_3\left\{\left[\lambda_3^2-\left(gH+3u^2\right)\right]\frac{\partial u_1}{\partial x}+\lambda_3\frac{\partial u_2}{\partial x}+\frac{\partial v_2}{\partial x}\right\}=0;$$

Спасибо за внимание